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Another Oscillation Criteria for First Order Non-Linear Differential Equation with Non-Monotone Delays

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Abstract: In this paper, the oscillatory behaviour of the first order non-linear delay differential equation $x'(t) + q(t)f(x(\tau(t))) = 0$, $t \geq t_0$, where q, τ ϵ C{[t_0, ∞),[0,∞)}, τ(t) < t, such that $\lim_{t\to\infty} \tau(t) = \infty$ is studied.

Keywords: Oscillation, Non-linear equation, Non-monotone delay.

1. Introduction

Consider the non-linear delay differential equation $x'(t) + q(t)f(x(\tau(t))) = 0$, $t \geq t_0$, (1.1)

where q, τ ϵ C{[t_0, ∞),[0,∞)}, τ(t) < t, such that $\lim_{t\to\infty} \tau(t) = \infty$, f ϵ C(R ,R) and xf(x) > 0 for x ≠ 0 and $M = \lim_{x\to 0} \frac{x}{f(x)} < \infty$. A solution of (1.1) is said to be oscillatory, if it has arbitrarily large zeroes. Otherwise, it is called non oscillatory.

Oscillation and delay phenomena appear in various models from real-world applications.

See, [8,9] for models from mathematical biology, where oscillation and delay actions may be formulated by means of cross-diffusion terms. In particular, the oscillation of first order delay differential equations has attracted the attention of many mathematicians.

Braverman and Karpuz [4] state that the classic condition for the oscillation of (1.1) in monotone delay case is

$$\lim_{t\to\infty} sup \int_{\tau(t)}^{t} q(u)\, du > 1, \tag{1.2}$$

is not applicable for the non-monotone case in general. This highlights the importance of obtaining necessary and/or sufficient conditions for the oscillation of the general form (1.1). In [1–3, 7] sharp oscillatory condition that improves (1.2) is obtained for a particular class of (1.1) with coefficients enjoying the slowly varying property. In this work, we obtain new sufficient criteria for the oscillation of (1.1) when τ is not assumed to be monotone and q need not to be of slowly varying type.

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We assume the existence of a non-decreasing continuous function h(t) such that $\tau(t) \leq h(t) < t$ for all $t \geq t_0$. Also ,let $\sigma(t) = \sup_{u \leq t} \tau(u)$, $t \geq t_0$. (1.3)

We also assume that $\lambda(\beta)$ is the smaller real root of the equation $\lambda = e^{\beta\lambda}$,

$$\Delta(u) = \frac{1 - u - \sqrt{1 - 2u - u^2}}{2}, \ 0 \leq u \leq \frac{1}{e}, \tag{1.4}$$

$$k = \liminf_{t \to \infty} \int_{\tau(t)}^{t} q(u)\, du \leq \frac{1}{e},$$

$$k^* = \liminf_{t \to \infty} \int_{h(t)}^{t} q(u)\, du ,$$

$$\rho = \begin{cases} 1, & k^* = 0 \\ \lambda(k^*) - \varepsilon, & k^* > 0, \varepsilon \in (0, \lambda(k^*)) \end{cases} \tag{1.5}$$

and $L^* = \lim_{t\to\infty} sup \int_{h(t)}^{t} q(u)\, du.$

Therefore, in this paper, we derive new iterative type oscillation criteria.

2. Preliminary Results

We use the following theorem and lemmas for our main results.

Theorem 2.1: (See [6], Theorem 2.1)

Suppose $\tau(t)$ is non – monotone or non – decreasing function , h(t) is non decreasing and $\tau(t) \leq h(t)$ for all $t \geq 0$.

Suppose that the function f in (1.1) satisfies the following condition,

$$\limsup_{x \to 0} \frac{x}{f(x)} = M, \ 0 \leq M < \infty \text{ and} \tag{2.1}$$

$$\text{if } \liminf_{t \to \infty} \int_{\tau(t)}^{t} q(s)ds > \frac{M}{e} \text{ holds,} \tag{2.2}$$

then all solutions of (1.1) oscillate.

Lemma 2.1: (See [6])

Let x(t) be an eventually positive solution of (1.1). Then,

$$\frac{x(h(t))}{x(t)} \geq \rho, \text{ for all sufficiently large t} \tag{2.3}$$

where ρ is defined by (1.4).

Lemma 2.2: (See [5])

Assume that x(t) is an eventually positive solution of (1.1) .Then ,

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$\lim_{t\to\infty} inf \frac{x(t)}{x(h(t))} \geq \Delta(k^*)$.

Lemma 2.3:

Assume that x(t) is a solution of (1.1) and n is a positive integer. Then,

$$x(h(t)) \geq x(t) + \sum_{i=1}^{n-1} \frac{x(h^i(t))}{M^i} Q_i^n(t) + \frac{x(h^n(t))}{M^n} \bar{Q}_n^n(t) , \quad (2.4)$$

where

$$Q_i^n(t) = \int_{h(t)}^{t} q(u_1) \int_{\tau(u_1)}^{h(t)} q(u_2) \ldots \int_{\tau(u_{i-1})}^{h^{i-1}(t)} q(u_i)\, du_i\, du_{i-1} \ldots du_1, \quad i = 1,2,\ldots n-1$$

and

$$\bar{Q}_n^n(t) = \int_{h(t)}^{t} q(u_1) \int_{\tau(u_1)}^{h(t)} q(u_2) \ldots \int_{\tau(u_{n-1})}^{h^{n-1}(t)} q(u_n)\, e^{\int_{\tau(u_n)}^{h^n(t)} \frac{q(u_{n+1})}{M} \frac{x(\tau(u_{n+1}))}{x(u_{n+1})} du_{n+1}} du_n\, du_{n-1} .. du_1.$$

Proof:

Let us first consider the linear delay differential equation of the form

$$x'(t) + q(t)(x(\tau(t))) = 0, \; t \geq t_0 > 0 , \quad (2.5)$$

where q and τ are continuous on [t_0 ,∞), $t_0 > 0$, q(t) ≥ 0 and $\lim_{t\to\infty} \tau(t) = \infty$ and f ϵ C(R,R),

If there exists a non-oscillatory solution x(t) of (1.1) , then -x(t) is also a solution of (1.1).The discussion is confined only to the case where the solution is eventually positive. Then there exists a $t_1 > t_0$ such that x(t), x(τ(t)) > 0, for all $t \geq t_1$.

Thus from (1.1), we have $x'(t) = - q(t) f (x(\tau(t))) \leq 0$ for all $t \geq t_1$. It means that x(t) is nonincreasing and has a limit ℓ ≥ 0 as t ⟶ ∞.

Now we claim that ℓ = 0. Condition (2.2) implies that

$\int_a^\infty q(t)\, dt = \infty$.

By [1, Theorem 3.1.5], we obtain $\lim_{t\to\infty} x(t) = 0$. Suppose M > 0.

Then, in view of (2.1) we can choose $t_2 > t_1$ so large that

$$f(x(t)) \geq \frac{1}{M} x(t) \text{ for } t \geq t_2. \quad (2.6)$$

On the other hand,we know from Lemma 2.1.1 [6], that

$$\liminf_{t\to\infty} \int_{h(t)}^{t} q(s)ds = \liminf_{t\to\infty} \int_{\tau(t)}^{t} q(s)ds. \quad (2.7)$$

Since τ(t) ≤ h(t) and z(t) is non increasing , by (1.1) we have

$$x'(t) + q(t) \left(\frac{1}{M} x(\tau(t))\right) \leq 0 , \; t \geq t_3 \quad (2.8)$$

Integrating (2.8) from h(t) to t ,we have

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$\int_{h(t)}^{t} x'(u_1)du_1 + \int_{h(t)}^{t} \frac{q(u_1)}{M} x(\tau(u_1))du_1 \leq 0,$

$$x(t) - x(h(t)) + \int_{h(t)}^{t} \frac{q(u_1)}{M} x(\tau(u_1))du_1 \leq 0. \quad (2.9)$$

Again integrating (2.8) from τ(v) to h(t), v ≤ t, it follows that

$x(h(t)) - x(\tau(v)) + \int_{\tau(v)}^{h(t)} \frac{q(u_2)}{M} x(\tau(u_2))du_2 \leq 0.$

Replacing v by u_1,

$$x(h(t)) - x(\tau(u_1)) + \int_{\tau(u_1)}^{h(t)} \frac{q(u_2)}{M} x(\tau(u_2))du_2 \leq 0. \quad (2.10)$$

Substituting (2.10) in (2.9)

$x(t) - x(h(t)) + \int_{h(t)}^{t} \frac{q(u_1)}{M} \{ x(h(t)) + \int_{\tau(u_1)}^{h(t)} \frac{q(u_2)}{M} x(\tau(u_2))du_2\} du_1 \leq 0$

$x(t) - x(h(t)) + \int_{h(t)}^{t} \frac{q(u_1)}{M} x(h(t))\, du_1 + \int_{h(t)}^{t} \frac{q(u_1)}{M} \int_{\tau(u_1)}^{h(t)} \frac{q(u_2)}{M} x(\tau(u_2))du_2\, du_1 \leq 0$

$$x(t) - x(h(t)) + \frac{x(h(t))}{M} \int_{h(t)}^{t} q(u_1)\, du_1 + \frac{1}{M^2} \int_{h(t)}^{t} q(u_1)\, q(u_2) x(\tau(u_2))du_2\, du_1 \leq 0 \quad (2.11)$$

We note that $\tau(v) \leq h^2(t)$, $for\ all\ v \leq h(t)$.

Here integrating (2.8) from τ(v) to $h^2(t)$,we obtain

We obtain

$\int_{\tau(v)}^{h^2(t)} x'(u_3)\, d\, u_3 + \int_{\tau(v)}^{h^2(t)} \frac{q(u_3)}{M} x(\tau(u_3))du_3 \leq 0,$

$x(h^2(t)) - x(\tau(v)) + \int_{\tau(v)}^{h^2(t)} \frac{q(u_3)}{M} x(\tau(u_3))du_3 \leq 0,$

$x(\tau(v)) \geq x(h^2(t)) + \int_{\tau(v)}^{h^2(t)} \frac{q(u_3)}{M} x(\tau(u_3))du_3,$

Replacing v by u_2,

$$x(\tau(u_2)) \geq x(h^2(t)) + \int_{\tau(u_2)}^{h^2(t)} \frac{q(u_3)}{M} x(\tau(u_3))du_3, \quad (2.12)$$

Substituting (2.12) in (2.11),

$$x(t) - x(h(t)) + \frac{x(h(t))}{M} \int_{h(t)}^{t} q(u_1)\, du_1 + \frac{1}{M^2} \int_{h(t)}^{t} q(u_1) \int_{\tau(u_1)}^{h(t)} q(u_2) \{ x(h^2(t)) + \int_{\tau(u_2)}^{h^2(t)} \frac{q(u_3)}{M} x(\tau(u_3))du_3\}\, du_2 \leq 0,$$

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$$x(t) - x(h(t)) + \frac{x(h(t))}{M}\int_{h(t)}^{t} q(u_1)\, du_1 + \frac{x(h^2(t))}{M^2} \int_{h(t)}^{t} q(u_1)\, du_1 \int_{\tau(u_1)}^{h(t)} q(u_2)\, du_2$$

$$+ \frac{1}{M^3}\int_{h(t)}^{t} q(u_1) \int_{\tau(u_1)}^{h(t)} q(u_2) \int_{\tau(u_2)}^{h^2(t)} q(u_3)\, x(\tau(u_3))\, du_3\, du_2 du_1 \le 0$$

$$x(h(t) \ge x(t) + \sum_{i=1}^{n-1} \frac{x(h^i(t)}{M^i} \int_{h(t)}^{t} q(u_1) \int_{\tau(u_1)}^{h(t)} q(u_2) \int_{\tau(u_2)}^{h^2(t)} q(u_3) \ldots du_i\, du_{i-1} \ldots du_1$$

$$+\frac{1}{M^n}\int_{h(t)}^{t} q(u_1) \int_{\tau(u_1)}^{h(t)} q(u_2) \ldots \int_{\tau(u_{n-1})}^{h^{n-1}(t)} q(u_n)\, x(\tau(u_n))\, du_n\, du_{n-1} \ldots du_1 \quad (2.13)$$

Dividing both sides of (2.8) by x(t) , where t is sufficiently large, and then integrating from $\tau(v)$ to $h^n(t)$,we get

$$\int_{\tau(v)}^{h^n(t)} \frac{x'(u_{n+1})}{x(u_{n+1})} du_{n+1} + \int_{\tau(v)}^{h^n(t)} \frac{q(u_{n+1})}{M} \frac{x(\tau(u_{n+1}))}{x(u_{n+1})}\, du_{n+1} \le 0$$

$$\ln \frac{x(\tau(v))}{x(h^n(t))} \ge \int_{\tau(v)}^{h^n(t)} \frac{q(u_{n+1})}{M} \frac{x(\tau(u_{n+1}))}{x(u_{n+1})}\, du_{n+1},$$

$$x(\tau(v) \ge x(h^n(t)) e^{\int_{\tau(v)}^{h^n(t)} \frac{q(u_{n+1})}{M} \frac{x(\tau(u_{n+1}))}{x(u_{n+1})} du_{n+1}}$$

Replacing v by u_n,

$$x(\tau(u_n) \ge x(h^n(t))\, e^{\frac{1}{M}\int_{\tau(u_n)}^{h^n(t)} q(u_{n+1}) \frac{x(\tau(u_{n+1}))}{x(u_{n+1})} du_{n+1}} \quad (2.14)$$

Substituting (2.14) in (2.13) ,

$$x(h(t) \ge x(t) + \sum_{i=1}^{n-1} \frac{x(h^i(t))}{M^i} \int_{h(t)}^{t} q(u_1) \int_{\tau(u_1)}^{h(t)} q(u_2) \int_{\tau(u_2)}^{h^2(t)} q(u_3) \ldots \int_{\tau(u_{i-1})}^{h^{i-1}(t)} q(u_i)\, du_i\, du_{i-1} \ldots du_1$$

$$+\frac{x(h^n(t))}{M^n}\int_{h(t)}^{t} q(u_1) \int_{\tau(u_1)}^{h(t)} q(u_2) \ldots \int_{\tau(u_{n-1})}^{h^{n-1}(t)} q(u_n)\, e^{\int_{\tau(u_n)}^{h^n(t)} \frac{q(u_{n+1})}{M} \frac{x(\tau(u_{n+1}))}{x(u_{n+1})} du_{n+1}}\, du_n\, du_{n-1} \ldots du_1$$

Let $Q_i^n(t) = \int_{h(t)}^{t} q(u_1) \int_{\tau(u_1)}^{h(t)} q(u_2) \ldots \int_{\tau(u_{i-1})}^{h^{i-1}(t)} q(u_i)\, du_i\, du_{i-1} \ldots du_1$, i = 1,2,…n-1

$$\bar{Q}_n^n(t) = \int_{h(t)}^{t} q(u_1) \int_{\tau(u_1)}^{h(t)} q(u_2) \ldots \int_{\tau(u_{n-1})}^{h^{n-1}(t)} q(u_n)\, e^{\int_{\tau(u_n)}^{h^n(t)} \frac{q(u_{n+1})}{M} \frac{x(\tau(u_{n+1}))}{x(u_{n+1})} du_{n+1}} du_n du_{n-1} \ldots du_1,$$

then we have ,

$$x(h(t) \ge x(t) + \sum_{i=1}^{n-1} \frac{x(h^i(t)}{M^i} Q_i^n(t) + \frac{x(h^n(t)}{M^n}\, \bar{Q}_n^n(t)$$ proving (2.4).

This completes the proof.

Lemma 2.4:

Assume that x(t) is an eventually positive solution of (1.1) and let $\phi = \liminf_{t\to\infty} \frac{x(t)}{x(h(t))}$.

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Then

$$\limsup_{t\to\infty}\left(\sum_{i=1}^{n}\frac{x\left(h^{i}(t)\right)}{x(h(t))}Q_i^n(\mathrm{t})\right)\le M^n[1-\phi].$$

Proof:

Using Lemma 2.1 and (2.4),we get

$$\mathrm{x(t)-x(h(t))}+\sum_{i=1}^{n}\frac{x(h^{i}(t))}{M^{i}}\ Q_i^n(\mathrm{t})\le 0.$$

Dividing by x(h(t)) ,

$$\frac{x(t)}{x(h(t))}-1+\sum_{i=1}^{n}\frac{x(h^{i}(t))}{M^{i}\mathrm{x(h(t))}}\ Q_i^n(\mathrm{t})\le 0.$$

$$\sum_{i=1}^{n}\frac{x(h^{i}(t))}{M^{i}\mathrm{x(h(t))}}\ Q_i^n(\mathrm{t})\le 1-\frac{x(t)}{x(h(t))}.$$

Consequently

$$\limsup_{t\to\infty}\left(\sum_{i=1}^{n}\frac{x(h^{i}(t))}{x(h(t))}Q_i^n(\mathrm{t})\right)\le M^n[1-\phi].$$

3. Main Results

Theorem 3.1:

Assume that n ϵ N and

$$\limsup_{t\to\infty}\left(\sum_{i=1}^{n}\left(\prod_{j=2}^{i}\psi(h^{j-1}(t))\,Q_i^n(\mathrm{t})\right)\right)>M^n[1-\Delta(\mathrm{k}^*)] \qquad (3.1)$$

where ψ(t) $\frac{1}{M-\int_{h(t)}^{t}q(u_1)\,exp\left(\int_{\tau(u_1)}^{h(t)}\frac{q(u_2))}{M-Q_1^1(u_2)}du_2\right)du_1}$, by convention we set $\prod_{j=2}^{1}\psi(h^j(t)=1$.

Then (1.1) is oscillatory.

Proof:

Let x(t) be a non – oscillatory solution of (1.1). We can choose x(t) to be eventually positive. Letting n=1 in (2.4), we obtain

$$\mathrm{x(h(t))}\ \ge\ \mathrm{x(t)}+\mathrm{x(h\ (t))}\int_{h(t)}^{t}\frac{q(u_1)}{M}\ e^{\int_{\tau(u_1)}^{h(t)}\frac{q(u_2)}{M}\frac{x(\tau(u_2))}{x(u_2)}du_2}du_1$$

$$\mathrm{x(h(t))}\ -\mathrm{x(h\ (t))}\int_{h(t)}^{t}\frac{q(u_1)}{M}\ e^{\int_{\tau(u_1)}^{h(t)}\frac{q(u_2)}{M}\frac{x(\tau(u_2))}{x(u_2)}du_2}du_1\ge \mathrm{x(t)}$$

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x(h(t)) (1 - $\int_{h(t)}^{t} \frac{q(u_1)}{M} \; e^{\int_{\tau(u_1)}^{h(t)} \frac{q(u_2)x(\tau(u_2))}{M \quad x(u_2)} du_2} du_1) \geq$ x(t)

Dividing by x(t),we get

$$\frac{x(h(t))}{x(t)} \geq \frac{M}{(M - \int_{h(t)}^{t} q(u_1) \; e^{\int_{\tau(u_1)}^{h(t)} \frac{q(u_2)x(\tau(u_2))}{M \quad x(u_2)} du_2} du_1)}. \tag{3.2}$$

Using Lemma 2.1 ,we obtain

$$\frac{x(h(t))}{x(t)} \geq \frac{M}{(M - \int_{h(t)}^{t} q(u_1) \; e^{\rho \int_{\tau(u_1)}^{h(t)} \frac{q(u_2)}{M} du_2} du_1)} = \frac{M}{M - Q_1^1(t)}.$$

Substituting in (3.2) ,

$$\frac{x(h(t))}{x(t)} \geq \frac{M}{(M - \int_{h(t)}^{t} q(u_1) \; e^{\int_{\tau(u_1)}^{h(t)} \frac{q(u_2)}{M} \frac{M}{M - Q_1^1(u_2)} du_2} du_1)} = M\,\psi(t) \tag{3.3}$$

But

$$\frac{x(h^i(t))}{x(h(t))} = \prod_{j=2}^{i} \frac{x(h^j(t))}{x(h^{j-1}(u_2))} \text{ ,for i = 2,3,…n.} \tag{3.4}$$

Then from (3.3) ,we get

$$\frac{x(h^i(t))}{x(h(t))} \geq \prod_{j=2}^{i} \; M^n \psi\,(h^{j-1}(t\,)) \tag{3.5}$$

Combining (3.5) with Lemmas 2.4 and using Lemma 2.2, we obtain

$\limsup_{t \to \infty} \left(\sum_{i=1}^{n} \left(\prod_{j=2}^{i} \psi(h^{j-1}(t))\, Q_i^n(t)\right)\right) \leq M^n$ [1 - Δ(k*)] which is a contradiction to (3.1). Hence the solutions of (1.1) are oscillatory.

Example:

Consider the non-linear delay differential equation

$$x'(t) + q(t)\, x(\tau(t))(|x(\tau(t)| + 8.68) = 0 \quad ,t \geq 2 \tag{3.6}$$

where

$\tau(t) = t - 1 - \alpha \ln(\cos^2(\nu\pi t))$

and

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$$q(t) = \begin{cases} 0\,, & t \in [c_r, d_r] \\ \beta(t - d_r)\cos(t - d_r)\,, & t \in [d_r, d_{r+1}] \\ \beta\,, & t \in [d_{r+1}, d_{r+6}] \\ \beta\left(1 - \frac{(t-(d_r+6)}{c_{r+1}-d_r-6}\right), & t \in [d_{r+6}, c_{r+1}], \end{cases}$$

$r \in N$, $0 \le c_r < d_r - 1 - \alpha$, $d_r + 6 < c_{r+1}$ such that $\lim_{r\to\infty} c_r = \infty$,

$\alpha = .00009$, $\beta = 0.45$, $\nu = 15000.$ *We* Choose $h(t) = \sigma(t)$, then

$t - 1 - \alpha \le \tau(t) \le h(t) = \sigma(t) \le t - 1$.

Therefore

$$0 \le \int_{\tau(c_r)}^{d_r} q(u)du \le \int_{c_r-1-\alpha}^{c_r} q(u)du = 0.$$

Then , $k = k^* = \lim_{t\to\infty} \inf \int_{\tau(t)}^{t} q(u)\,du = 0$ and hence $\rho = 1$.

$$M = \lim_{x\to 0} \sup \frac{x}{f(x)} = \lim_{x\to 0} \sup \frac{x}{x(|x|+8.68)} = 0.1152$$

Clearly

$$Q_1^2(d_r+6) = \int_{h(d_r+6)}^{d_r+6} q(u_1)\,du_1$$

$$\ge \int_{d_r+5}^{d_r+6} q(u_1)\,du_1$$

$$= \int_{d_r+5}^{d_r+6} \beta\,du_1$$

$$= 0.45.$$

$$Q_2^2(d_r+6) = \int_{h(d_r+6)}^{d_r+6} q(u_1)\,du_1 \int_{\tau(u_1)}^{h(d_r+6)} q(u_2)\,du_1 e^{\rho\int_{\tau(u_2)}^{h^2(d_r+6)} \frac{q(u_3)}{M} du_3}\,du_2 du_1$$

$$\ge \int_{d_r+5}^{d_r+6} q(u_1)\,du_1 \int_{u_1-1}^{d_r+5-\alpha} q(u_2)\,du_1 e^{\int_{u_2-1}^{d_r+4-2\alpha} \frac{q(u_3)}{M} du_3}\,du_2 du_1$$

$$= 0.004.$$

Also for $d_r + 3 \le \nu \le d_r + 4 - \alpha$,

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$$Q_1^1(\nu) = \int_{h(\nu)}^{\nu} q(u_1)\, e^{\rho \int_{\tau(u_1)}^{h(\nu)} \frac{q(u_2)}{M} du_2}\, du_1$$

$$\geq \int_{\nu-1}^{\nu} q(u_1)\, e^{\int_{u_1-1}^{\nu-1-\alpha} \frac{q(u_2)}{M} du_2}\, du_1$$

$$= \int_{\nu-1}^{\nu} \beta\, e^{\int_{u_1-1}^{\nu-1-\alpha} \frac{\beta}{M} du_2}\, du_1 > 3.159.$$

Consequently,

$$\Psi(\mathrm{h}(d_r+6)) = \frac{M}{M - \int_{h^2(d_r+6)}^{h(d_r+6)} \frac{q(u_1)}{M}\, e^{\int_{h(u_1)}^{h^2(d_r+6)} \frac{q(u_2)}{M-Q_1^1(u_2)} du_2}\, du_1}$$

$$\geq \frac{M}{M - \int_{(d_r+4)}^{(d_r+5-\alpha)} \frac{q(u_1)}{M}\, e^{\int_{(u_1-1)}^{(d_r+4-2\alpha)} \frac{q(u_2)}{M-Q_1^1(u_2)} du_2}\, du_1}$$

$$> -0.294.$$

Thus,

$$\limsup_{t\to\infty}\left(\sum_{i=1}^{2}\left(\prod_{j=2}^{2}\psi(h^{j-1}(t))\, Q_2^2(t)\right)\right) = 0.4488.$$

And

$$M^2\,[1 - \Delta(k^*)] = 0.0132.$$

Thus proving (3.1) that

$$\limsup_{t\to\infty}\left(\sum_{i=1}^{2}\left(\prod_{j=2}^{2}\psi(h^{j-1}(t))\, Q_2^2(t)\right)\right) = Q_1^2(d_r+6) + \Psi(\mathrm{h}(d_r+6))Q_1^2(d_r+6)$$

$$= 0.4488$$

$$> M^2\,[1 - \Delta(k^*)]$$

$$= 0.0132.$$

It follows from (3.1) that the equation is oscillatory.

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