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Some Characterizations based on Generalized Order Statistics from Weibull-Family of Life Distributions

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Abstract:
Characterizations of probability distributions based on recurrence relations for single and product moments of generalized order statistics has attracted the attention of many researchers. We present here, characterizations of Weibull -family of life distributions based on generalized order statistics.

Characterizations of distributions based on recurrence relations for single and product moments of generalized order statistics have been investigated quite extensively in the literature involving ordered random variables. As generalized order statistics (GOS) provide a unifying approach to models of ordered random variables.

Keywords: Generalized order statistics; Order statistics; Records ; Single and product moments; Recurrence relations ; Characterizations.

1. Introduction

The primary objective of this article is to offer characterizations for absolutely continuous distributions, utilizing recurrence relations for both single and product moments of GOS. It is important to note that our study does not seek to extend all characterization results in this context. However, our discoveries and mathematical approaches not only contribute novel characterization results for diverse models of ordered random variables but also hold potential applications in various aspects of GOS.

Bourguignon et al. (2014) introduced a new method of adding a parameter into a family of distributions. The resulting distribution, known as the Weibull generated distribution, includes

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the original distribution as a special case and gives more flexibility to model various types of data.

Let $G(x,\xi)$ be a continuous baseline distribution with density $g(x,\xi)$ depends on a parameter vector.

The Cumulative density function (cdf) of the Weibull-distribution

$$F(x,\alpha,\beta)=1-e^{-\alpha x^{\beta}}; \qquad x,\alpha,\beta>0.$$

The cdf of the Weibull– G family is given by

$$F(x,\alpha,\beta,\xi)=\int_0^{\left(\frac{G(x)}{\bar{G}(x)}\right)}\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}dx=1-e^{-\alpha\left(\frac{G(x)}{\bar{G}(x)}\right)^{\beta}}.$$

The probability density function (pdf) of the Weibull– G family is given by

$$\bar{F}(x)=e^{-\alpha\left(\frac{G(x)}{\bar{G}(x)}\right)^{\beta}}; \qquad x,\beta,\alpha>0, \tag{1}$$

where α and β are the scale and shape parameters, respectively. The probability density function (*pdf*) corresponding to $\bar{F}(x)$ are:

$$f(x)=\alpha\beta g(x)\frac{G(x)^{\beta-1}}{\bar{G}(x)^{\beta+1}}e^{-\alpha\left(\frac{G(x)}{\bar{G}(x)}\right)^{\beta}}; \qquad x,\beta,\alpha>0, \tag{2}$$

where $\bar{G}(x)$ as

$$\bar{G}(x)=\mathrm{e}^{-\lambda(x)}; \qquad x\ge 0,\ \theta>0, \tag{3}$$

where $\lambda(x)$ is a non-negative, continuous, monotone increasing, differentiable function of x such that $\lambda(x)\to 0$ as $x\to 0^{+}$ and $\lambda(x)\to\infty$ as $x\to\infty$ (see Mahmoud and Ghazal (2012)).

Substituting from (3) in (1), we get

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$$\bar{F}(\mathrm{x}) = e^{-\alpha\left(e^{\lambda(x)}-1\right)^{\beta}} \qquad x \ge 0 \tag{4}$$

The pdf corresponding to $\bar{F}(x)$

$$f(x) = \alpha\beta\lambda'(x)e^{\lambda(x)}\left(e^{\lambda(x)}-1\right)^{\beta-1}\mathrm{e}^{-\alpha\left(e^{\lambda(x)}-1\right)^{\beta}} \qquad x \ge 0 \tag{5}$$

The family of distributions in (5), is called Weibull-family of life distributions.
Now in view of (4) and (5), we get

$$\bar{\mathrm{F}}(x) = \frac{f(x)}{\alpha\beta\lambda'(x)}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}\lambda^{v}(x)}{v!} \tag{6}$$

Then distributions can be obtained from (5), as illustrated.

Table (1) examples of *pdf* (5)

$\lambda(x)$	*pdf*	Distribution
λx; $\lambda > 0$ and $x \ge 0$	$\alpha\beta\lambda e^{\lambda x}\left(e^{\lambda x}-1\right)^{\beta-1}e^{-\alpha\left(e^{\lambda x}-1\right)^{\beta}}$	Weibull-exponential
$ax+\frac{b}{2}x^2$; a, x and $b > 0$	$ab\alpha\beta e^{ax+\frac{b}{2}x^2}\left(e^{ax+\frac{b}{2}x^2}-1\right)^{\beta-1}\mathrm{e}^{-\alpha\left(e^{ax+\frac{b}{2}x^2}-1\right)^{\beta}}$	Weibull-linear failure rate
λx^{θ}; $\lambda,\theta > 0$ and $x \ge 0$	$\alpha\beta\lambda\theta x^{\theta-1}e^{\lambda x^{\theta}}\left(e^{\lambda x^{\theta}}-1\right)^{\beta-1}\mathrm{e}^{-\alpha\left(e^{\lambda x^{\theta}}-1\right)^{\beta}}$	Weibull-Weibull
λx^2 $\lambda \succ 0$ and $x \ge 0$	$2\alpha\beta\lambda x e^{\lambda x^2}\left(e^{\lambda x^2}-1\right)^{\beta-1}e^{-\alpha\left(e^{\lambda x^2}-1\right)^{\beta}}$	Weibull-Rayleigh
$\lambda x^{\theta}e^{\gamma x}$ $\lambda,\theta,\gamma > 0$ and $x \ge 0$	$\alpha\beta 2\lambda\theta\gamma e^{2\gamma x}x^{2\theta-1}\mathrm{e}^{\lambda x^{\theta}e^{\gamma x}}\left(e^{\lambda x^{\theta}e^{\gamma x}}-1\right)^{\beta-1}\mathrm{e}^{-\alpha\left(e^{\lambda x^{\theta}e^{\gamma x}}-1\right)^{\beta}}$	Modified - Weibull

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$\frac{a}{c}\left(e^{cx}-1\right)$ $c\geq 0$ and $x,a>0$	$\alpha\beta a e^{cx} e^{\frac{a}{c}\left(e^{cx}-1\right)}\left(e^{\frac{a}{c}\left(e^{cx}-1\right)}-1\right)^{\beta-1} \mathrm{e}^{-\alpha\left(e^{\frac{a}{c}\left(e^{cx}-1\right)}-1\right)^{\beta}}$	Weibull - Gompertz
$a\ln\left(1+x^{b}\right)$; $a,b,x>0$	$\frac{\alpha\beta abx^{b-1}}{\left(1+x^{b}\right)} e^{a\ln\left(1+x^{b}\right)}\left(e^{a\ln\left(1+x^{b}\right)}-1\right)^{\beta-1} \mathrm{e}^{-\alpha\left(e^{a\ln\left(1+x^{b}\right)}-1\right)^{\beta}}$	Weibull – Burr type XII
$a\ln\left(1+bx\right)$; $a,b,x>0$	$\frac{ab\alpha\beta}{\left(1+bx\right)} e^{a\ln(1+bx)}\left(e^{a\ln(1+bx)}-1\right)^{\beta-1} \mathrm{e}^{-\alpha\left(e^{a\ln(1+bx)}-1\right)^{\beta}}$	Weibull – Lomax
$a\ln\left(1+x\right)$ $a,x>0$	$\frac{a\alpha\beta}{\left(1+x\right)} e^{a\ln(1+x)}\left(e^{a\ln(1+x)}-1\right)^{\beta-1} \mathrm{e}^{-\alpha\left(e^{a\ln(1+x)}-1\right)^{\beta}}$	Weibull – Pareto

Kamps (1995) introduced the concept of generalized order statistics (GOS), encompassing a range of order models for random variables. For simplicity, let F consistently represent an absolutely continuous distribution function with a corresponding density function *f*.

The random variables $X\left(1,n,\tilde{m},k\right),\ldots,X\left(n,n,\tilde{m},k\right)$ are called generalized order statistics based on F, if their joint *pdf* of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_{j}\right)\left(\prod_{i=1}^{n-1}\left[\bar{F}\left(x_{i}\right)\right]^{m_{i}} f\left(x_{i}\right)\right)\left[\bar{F}\left(x_{n}\right)\right]^{k-1} f\left(x_{n}\right),$$

for $F^{-1}\left(0\right)<x_{1}\leq x_{2}\leq\ldots\leq x_{n}<F^{-1}\left(1\right)$. with parameters $n\in N,\ k\geq 2,\ k>0,$

$\tilde{m}=\left(m_{1},m_{2},\ldots,m_{n-1}\right)\in R^{n-1}, M_{r}=\sum_{i=r}^{n-1} m_{i}$, such that $\gamma_{r}=k+n-r+M_{r}>0,$ for all

$r\in\left\{1,2,\ldots,n-1\right\}$. For $\gamma_{i}\neq\gamma_{j}$ for all $i,j\in\left(1,2,\ldots,n-1\right)$ the pdf of $X\left(r,n,\tilde{m},k\right)$ is given by Cramer and Kamps (2000) in the following way

$$f_{X\left(r,n,\tilde{m},k\right)}\left(x\right)=C_{r-1} f\left(x\right)\sum_{i=1}^{r} a_{i}\left(r\right)\left[\bar{F}\left(x\right)\right]^{\gamma_{i}-1}. \qquad (7)$$

The joint *pdf* of $X\left(r,n,\tilde{m},k\right)$ and $X\left(s,n,\tilde{m},k\right)$, $1\leq r<s\leq n$ is given as

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$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y)=C_{s-1}\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}\left(\sum_{i=1}^{r}a_i^{(r)}\left[\bar{F}(x)\right]^{\gamma_i}\right)\frac{f(x)f(y)}{\bar{F}(x)\bar{F}(y)},$$

$$x<y \quad (8)$$

Where

$$a_i(r)=\prod_{\substack{j=1\\ j\neq i}}^{r}\frac{1}{\gamma_j-\gamma_i}, \qquad 1\le i\le n$$

and $a_i^{(r)}(s)=\prod_{\substack{j=r+1\\ j\neq i}}^{s}\frac{1}{\gamma_j-\gamma_i}, \qquad r+1\le i\le s\le n.$

It may be noted that for $m_1=m_2=\ldots=m_{n-1}=m\neq-1$

$$a_i(r)=\frac{(-1)^{r-i}\binom{r-1}{r-i}}{(m+1)^{r-1}(r-1)!}, \quad (9)$$

and $$a_i^{(r)}(s)=\frac{(-1)^{s-i}\binom{s-r-1}{s-i}}{(m+1)^{s-r-1}(s-r-1)!}, \quad (10)$$

Therefore *pdf* of $X(r,n,\tilde{m},k)$ given in (7) reduces to

$$f_{X(r,n,m,k)}(x)=\frac{C_{r-1}}{\Gamma(r)}\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right], \quad x\in\chi \quad (11)$$

and joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k), 1\le r<s\le n$ is given in (8) reduced to

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y)=\frac{C_{s-1}}{(r-1)!\Gamma(s-r-1)!}\left[\bar{F}(x)\right]^{m}f(x)g_m^{r-1}\left[F(x)\right]$$

$$\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s-1}f(y), \quad x<y, \quad (12)$$

Where

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i, \quad \gamma_i=k+(n-i)(m+1),$$

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$$h_m(x)=\begin{cases}\frac{-(1-x)^{m+1}}{m+1}, & m=-1\\ \log\left(\frac{1}{1-x}\right), & m\neq -1\end{cases}$$

and

$$g_m(x)=h_m(x)-h_m(1),\quad x\in[0,\ 1).$$

We shall also take $X(0,n,m,k)=0$. If m = 0, k = 1, then $X(r,n,m,k)$ reduces to the $(n-r+1)^{th}$ order statistics, $X_{n-r+1:n}$ from the sample $X_1, X_2, ..., X_n$ and when m = -1, then $X(r,n,m,k)$ reduces to the k^{th} record value (Pawlas and Szynal [2001]).

Numerous authors have incorporated generalized order statistics (GOS) into their research, including Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000, 2016), Pawlas and Szynal (2001), Ahmed (2007), Ahmed and Fawzy (2007), Khan et al. (2007), AL-Hussaini et al. (2005), Kumar (2011), Abdul-Moniem (2019), Nagwa (2020), A-Rahman et. al. [2023,2024,2025]. and Alimohammadi (2022). These works explore various characterizations through conditional events of generalized order statistics.

In this paper, we present a characterization of the Weibull-Family distribution, employing a recurrence relation for both single and product moments of generalized order statistics (GOS).

2. Characterization Based on recurrence relation for single moments of gos

Theorem 2.1 Let X be a non-negative random variable having an absolutely continuous distribution function F(x) with F(0) = 0 and 0 < F(x) < 1 for all x > 0, then

$$E\left[X^j(r,n,m,k)\right]-E\left[X^j(r-1,n,m,k)\right]=\frac{j}{\alpha\beta\gamma_r}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}\,E\left[\frac{X^{j-1}(r,n,m,k)\lambda^v(X(r,n,m,k))}{\lambda'(X(r,n,\tilde{m},k))}\right] \quad (13)$$

if and only if $\bar{F}(y)=e^{-\alpha\left(e^{\lambda(y)}-1\right)^{\beta}}$

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Proof 2.1.

(i) Necessity

Building upon Lemma 2.3 (Athar and Islam [2004]) that

$$E\left[\xi\left\{X\left(r,n,\tilde{m},k\right)\right\}\right]-E\left[\xi\left\{X\left(r-1,n,\tilde{m},k\right)\right\}\right]=C_{r-2}\int_{\theta}^{\beta}\xi'(x)\sum_{i=1}^{r}a_i(x)\left[\bar{F}(x)\right]^{\gamma_r}dx$$

If we let $\xi(x)=x^j$, then

$$E\left[X^j\left(r,n,\tilde{m},k\right)\right]-E\left[X^j\left(r-1,n,\tilde{m},k\right)\right]=jC_{r-2}\int_{\theta}^{\beta}x^{j-1}\sum_{i=1}^{r}a_i(x)\left[\bar{F}(x)\right]^{\gamma_r}dx \qquad (14)$$

On using (6) in (14), we get

$$E\left[X^j\left(r,n,\tilde{m},k\right)\right]-E\left[X^j\left(r-1,n,\tilde{m},k\right)\right]=\frac{jC_{r-1}}{\alpha\beta\gamma_r}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}\int_0^{\infty}\frac{x^{j-1}\lambda^v(x)}{\lambda'(x)}\sum_{i=1}^{r}a_i(x)\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)dx$$

The simplification process results in Eq (13).

(ii) Sufficiency

On the other hand if the recurrence relation in Eq (13) is satisfied, then by using Eq (11), we have

$$\frac{C_{r-1}}{(r-1)!}\int_0^{\infty}x^j\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx-\frac{C_{r-2}}{(r-2)!}\int_0^{\infty}x^j\left[\bar{F}(x)\right]^{\gamma_{r-1}-1}f(x)g_m^{r-2}\left[F(x)\right]dx$$
$$=\frac{jC_{r-1}}{\alpha\beta(r-1)!\gamma_r}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}\int_0^{\infty}\frac{x^{j-1}\lambda^v(x)}{\lambda'(x)}\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx$$

By integrating the first term on the left-hand side using parts, we obtain:

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{\infty}x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r}g_m^{r-1}\left[F(x)\right]dx$$
$$=\frac{jC_{r-1}}{\alpha\beta(r-1)!\gamma_r}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}\int_0^{\infty}\frac{x^{j-1}\lambda^v(x)}{\lambda'(x)}\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx$$

This implies that

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{\infty}x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]\left\{\bar{F}(x)-\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}\frac{\lambda^v(x)}{\alpha\beta\lambda'(x)}f(x)\right\}dx=0 \qquad (15)$$

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Now applying a generalization of the Muntz-Szasz theorem (Hwang and Lin [1984]) to (15), we get

$$\bar{F}(x)-\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}}{v!}\frac{\lambda^{v}(x)}{\alpha\beta\lambda'(x)}f(x)=0,$$

hence,

$$\bar{F}(x)=\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}}{v!}\frac{\lambda^{v}(x)}{\alpha\beta\lambda'(x)}f(x)$$

$$=\sum_{u=0}^{\infty}\binom{1-\beta}{u}(-1)^{u}\frac{e^{-(\beta+u)\lambda(x)}}{\alpha\beta\lambda'(x)}f(x)$$

This implies that

$$\bar{F}(x)=\frac{e^{-\lambda(x)}f(x)}{\alpha\beta\lambda'(x)}\left(e^{\lambda(x)}-1\right)^{1-\beta}$$

Integrating both side from 0 to y, we get

$$\int_{0}^{y}\frac{f(x)}{\bar{F}(x)}dx=\alpha\beta\int_{0}^{y}\lambda'(x)e^{\lambda(x)}\left(e^{\lambda(x)}-1\right)^{\beta-1}dx$$

This implies that

$$-\ln\left[\bar{F}(y)\right]=\alpha\left(e^{\lambda(y)}-1\right)^{\beta}$$

which prove that

$$\bar{F}(y)=e^{-\alpha\left(e^{\lambda(y)}-1\right)^{\beta}}\quad;\qquad y\geq 0$$

Corollary 2.2 for $m_1=m_2=\ldots=m_{n-1}=m\neq-1$,the recurrence relations for single moment of gos for W−family of life distributions is given as

$$E\left[X^{j}(r,n,m,k)\right]-E\left[X^{j}(r-1,n,m,k)\right]=\frac{j}{\alpha\beta\gamma_r}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}}{v!}$$
$$E\left[\frac{X^{j-1}(r,n,m,k)\lambda^{v}(X(r,n,m,k))}{\lambda'(X(r,n,\tilde{m},k))}\right]$$

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(16)

Proof. This can easy be deduced from (13) in view of the relation (9).

Note that: By considering and analyzing generalized order statistics (GOS), we can derive certain characterizations for the Weibull-Weibull distribution. $\lambda(x)=x^{\theta}$ in (3), established by A-Rahman et. al. (2023).

Remark 2.1 Putting *m = 0, k = 1* in Theorem 2.1., we obtain recurrence relations for single moments of order statistics as

$$E\left[X_{r:n}^{j}\right]-E\left[X_{r-1:n}^{j}\right]=\frac{j}{\alpha\beta(n-r+1)}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}}{v!}E\left[\frac{X_{r:n}^{j-1}\lambda^{v}(X_{r:n})}{\lambda'(X_{r:n})}\right]$$

(17)

Remark 2.2 Setting *m* = −*1, k = 1* in Theorem 2.1.,we obtain the recurrence relations of upper record values as

$$E\left[X_{(r:n,-1,1)}^{j}\right]-E\left[X_{(r-1:n,-1,1)}^{j}\right]=\frac{j}{\alpha\beta(n-r+1)}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}}{v!}E\left[\frac{X_{(r:n,-1,1)}^{j-1}\lambda^{v}\left(X_{(r:n,-1,1)}\right)}{\lambda'\left(X_{(r:n,-1,1)}\right)}\right]\quad(18)$$

3. Characterization based on recurrence relation for product moments of *gos*

Theorem 3.1 Let X be a non-negative random variable having an absolutely continuous distribution function F (x) with F (0) = 0 and 0 < F (x) < 1 for all $xy>0$, then

$$E\left[X^{i}(r,n,m,k).X^{j}(s,n,m,k)\right]-E\left[X^{i}(r,n,m,k).X^{j}(s-1,n,m,k)\right]$$
$$=\frac{j}{\alpha\beta\gamma_{r}}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}}{v!}E\left[\frac{X^{j-1}(r,n,m,k)\lambda^{v}(X(r,n,m,k))}{\lambda'(X(r,n,\tilde{m},k))}\right]$$

(19)

if and only if $\bar{F}(y)=e^{-\alpha\left(e^{\lambda(y)}-1\right)^{\beta}}$

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Proof

(i) Necessity

Building upon Lemma 3.2 (Athar and Islam [7]) that

$$E\left[\xi\left\{X\left(r,n,m,k\right).X\left(s,n,m,k\right)\right\}\right]-E\left[\xi\left\{X\left(r,n,m,k\right).X\left(s-1,n,m,k\right)\right\}\right]=$$
$$\frac{C_{s-2}}{(r-1)!(s-r-1)!}\int_{\theta}^{\beta}\int_{x}^{\beta}\frac{\partial}{\partial y}\xi\left(x,y\right)\left[\bar{F}\left(x\right)\right]^{m}f\left(x\right)g_{m}^{r-1}\left[F\left(x\right)\right]$$
$$\left[h_{m}\left(F\left(y\right)\right)-h_{m}\left(F\left(x\right)\right)\right]^{s-r-1}\left[\bar{F}\left(y\right)\right]^{\gamma_{s}}dydx$$

where $\xi\left(x,y\right)=\xi_{1}\left(x\right)\xi_{2}\left(y\right)$

If let $\xi\left(x,y\right)=x^{i}y^{j}$, then

$$E\left[X^{i}\left(r,n,m,k\right).X^{j}\left(s,n,m,k\right)\right]-E\left[X^{i}\left(r,n,m,k\right).X^{j}\left(s-1,n,m,k\right)\right]=$$
$$\frac{C_{s-2}}{(r-1)!(s-r-1)!}\int_{\theta}^{\beta}\int_{x}^{\beta}x^{i}y^{j-1}\left[\bar{F}\left(x\right)\right]^{m}f\left(x\right)g_{m}^{r-1}\left[F\left(x\right)\right]$$
$$\left[h_{m}\left(F\left(y\right)\right)-h_{m}\left(F\left(x\right)\right)\right]^{s-r-1}\left[\bar{F}\left(y\right)\right]^{\gamma_{s}}dydx$$

On employing equation (6), it obtain:

$$E\left[X^{i}\left(r,n,m,k\right).X^{j}\left(s,n,m,k\right)\right]-E\left[X^{i}\left(r,n,m,k\right).X^{j}\left(s-1,n,m,k\right)\right]=$$
$$\frac{C_{s-2}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}(-1)^{u+v}\left(\beta+u\right)^{v}\lambda^{v}}{\alpha\beta\left(r-1\right)!\left(s-r-1\right)!v!}\int_{0}^{\infty}\int_{x}^{\infty}x^{i}\frac{y^{j-1}\lambda^{v}\left(y\right)}{\lambda'\left(y\right)}\left[\bar{F}\left(x\right)\right]^{m}f\left(x\right)g_{m}^{r-1}\left[F\left(x\right)\right]$$
$$\left[h_{m}\left(F\left(y\right)\right)-h_{m}\left(F\left(x\right)\right)\right]^{s-r-1}\left[\bar{F}\left(y\right)\right]^{\gamma_{s}-1}f(y)dydx$$

This, following simplification, results in equation (19).

(ii) Sufficiency

If the recurrence relation in equation (19) is satisfied, then by using equation (12), we have

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$$\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^j\left[\bar{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1}$$

$$\left[\bar{F}(y)\right]^{\gamma_s-1} f(y)dydx-\frac{C_{s-2}}{(r-1)!(s-r-2)!}\int_0^\infty\int_x^\infty x^i y^j\left[\bar{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$

$$\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-2}\left[\bar{F}(y)\right]^{\gamma_{s-1}-1} f(y)dydx=\frac{jC_{s-1}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}(-1)^{u+v}(\beta+u)^v\lambda^v}{\alpha\beta\gamma_s\lambda(r-1)!(s-r-1)!v!}$$

$$\int_0^\infty\int_x^\infty x^i\frac{y^{j-1}\lambda^v(y)}{\lambda'(y)}\left[\bar{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s-1} f(y)dydx$$

By integrating the first term on the left-hand side using parts, it obtain:

$$\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\theta\int_x^\theta x^i y^{j-1}\left[\bar{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1}$$

$$\left[\bar{F}(y)\right]^{\gamma_s}dydx=\frac{jC_{s-1}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}(-1)^{u+v}(\beta+u)^v\lambda^v}{\alpha\beta\gamma_s\lambda(r-1)!(s-r-1)!v!}\int_0^\theta\int_x^\theta x^i\frac{y^{j-1}\lambda^v(y)}{\lambda'(y)}\left[\bar{F}(x)\right]^m$$

$$f(x)g_m^{r-1}\left[F(x)\right]\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1} f(y)dydx$$

This implies that

$$\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\theta\int_x^\theta x^i y^{j-1}\left[\bar{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1}$$
$$\left[\bar{F}(y)\right]^{\gamma_s-1}\left\{\bar{F}(y)-\frac{1}{\alpha\theta\beta v!}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}(-1)^{u+v}(\beta+u)^v\lambda^v f(y)\right\}dydx=0 \quad (20)$$

Now applying a generalization of the Muntz-Szasz theorem (Hwang and Lin [1984]) to (20), we get

$$\bar{F}(y)-\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}\frac{\lambda^v(y)}{\alpha\beta\lambda'(y)}f(y)=0,$$

hence,

$$\bar{F}(y)=\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}\frac{\lambda^v(y)}{\alpha\beta\lambda'(y)}f(y)$$
$$=\sum_{u=0}^{\infty}\binom{1-\beta}{u}(-1)^u\frac{e^{-(\beta+u)\lambda(y)}}{\alpha\beta\lambda'(y)}f(y)$$

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This implies that

$$\bar{F}(y)=\frac{e^{-\lambda(y)}f(y)}{\alpha\beta\lambda'(y)}\left(e^{\lambda(y)}-1\right)^{1-\beta}$$

Integrating both side from 0 to y, we get

$$\int_0^y \frac{f(x)}{\bar{F}(x)}dx=\alpha\beta\int_0^y \lambda'(x)e^{\lambda(x)}\left(e^{\lambda(x)}-1\right)^{\beta-1}dx$$

This implies that

$$-\ln\left[\bar{F}(y)\right]=\alpha\left(e^{\lambda(y)}-1\right)^{\beta}$$

which prove that

$$\bar{F}(y)=e^{-\alpha\left(e^{\lambda(y)}-1\right)^{\beta}}\quad;\qquad y\geq 0$$

Remark 3.1 Putting $m=0,\ k=1$ in (19), it obtain recurrence relations for product moments of order statistics as

$$E\left[X_{r,s:n}^{i,j}\right]-E\left[X_{r,s-1:n}^{i,j}\right]==\frac{j}{\alpha\beta(n-r+1)}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}}{v!}E\left[\frac{X_{r,s:n}^{i,j}\lambda^{v}\left(X_{r,s:n}^{i,j}\right)}{\lambda'\left(X_{r,s:n}^{i,j}\right)}\right]\tag{21}$$

Remark 3.2 Setting $m=-1$ in (19), it obtain the recurrence relations for product moments of k^{th} record values as

$$E\left[\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^j\right]-E\left[\left(X_r^{(k)}\right)^i\left(X_{s-1}^{(k)}\right)^j\right]$$
$$=\frac{j}{k\alpha\beta}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}}{v!}E\left[\frac{\left(X_r^{(k)}\right)^i\lambda^{v}\left(X_{s-1}^{(k)}\right)^j}{\lambda'\left(X_r^{(k)}\right)^i\left(X_{s-1}^{(k)}\right)^j}\right]\tag{22}$$

4. Conclusion

In this paper we have studied the characterizations of a family called Weibull-Family of Life distributions based on recurrence relations for single and product moments of generalized order

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