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Some Integral Inequalities for Differentiable *S*-Convex and (*H*, *M*)-Convex Functions Through Generalized Caputo-Type Derivatives

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Abstract
In this work we obtain integral inequalities of the Hermite-Hadamard type, using generalized derivatives of the Caputo type. Throughout the work, we see that several results reported in the literature are particular cases of those presented here.

Keywords: Integral inequalities, (*h*, *m*)-convex functions, Generalized Caputo type derivatives
AMS Subject Classification (2020): 26A33, 26D10, 47A63

1 Introduction

In Mathematics, the notion of convex function plays a very prominent role, due to its multiple appli-cations and its theoretical overlaps with various other areas of science (see [23, 31-34] for more information).

One of the most important inequalities for convex functions is the well-known Hermite-Hadamard inequality (see [16, 17]):

$$\psi\left(\frac{\nu_1+\nu_2}{2}\right)\leq\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\psi(x)dx\leq\frac{\psi(\nu_1)+\psi(\nu_2)}{2}.$$

This inequality holds for any convex ψ function on the interval $[\nu_1, \nu_2]$. Gives an estimate of the mean value of a convex function. Various extensions and generalizations can be consulted in [1, 2, 4, 5, 14, 15, 20, 24, 27] and references therein.

In [4] we presented the following definitions.

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Definition 1. *Let* h: $[0, 1]\to R$ *be a nonnegative function,* h /=0 *and* ψ: $I=[0,+\infty)\to[0,+\infty)$. *If inequality*

$$\psi\,(\tau\xi+m(1-\tau)\varsigma)\leq h^{s}(\tau)\psi(\xi)+m(1-h^{s}(\tau))\psi\,(\varsigma) \qquad (1)$$

is fulfilled for all ξ, $\varsigma\in I$ *and* $\tau\in[0, 1]$, *where* $m\in[0, 1]$, $s\in[-1, 1]$. *Then a function* ψ *is called a* (h, m)*-convex modified of the first type on* I.

Definition 2. *Let* h: $[0, 1]\to R$ *nonnegative functions,* h /=0 *and* ψ: $I=[0,+\infty)\to[0,+\infty)$. *If inequality*

$$\psi\,(\tau\xi+m(1-\tau)\varsigma)\leq h^{s}(\tau)\psi(\xi)+m(1-h(\tau))^{s}\psi\,(\varsigma)\ (2)$$

is fulfilled for all ξ, $\varsigma\in I$ *and* $\tau\in[0, 1]$, *where* $m\in[0, 1]$, $s\in[-1, 1]$. *Then a function* ψ *is called a* (h, m)*-convex modified of the second type on* I.

Remark 3. *From Definitions 1 and 2 we can define* $N^{s}{}_{h,m}[a, b]$, *where* $a, b\in[0,+\infty)$, *as the set of functions* (h, m)*-convex modified, for which* $\psi(a)\geq 0$, *characterized by the triple* $(h(\tau), m, s)$. *Note that if:*

1. $(h(\tau), 0, 0)$ *we have the increasing functions ([8]).*
2. $(\tau, 0, s)$ *we have the s-starshaped functions ([8]).*
3. $(\tau, 0, 1)$ *we have the starshaped functions ([8]).*
4. $(\tau, 1, 1)$ *then* ψ *is a convex function on* $[0,+\infty)$ *([8]).*
5. $(1, 1, s)$ *then* ψ *is a P-convex function on* $[0,+\infty)$ *([11]).*
6. $(\tau, m, 1)$ *then* ψ *is a m-convex function on* $[0,+\infty)$ *([28]).*
7. $(\tau, 1, s)$ $s\in(0, 1]$ *then* ψ *is a s-convex function on* $[0,+\infty)$ *([7, 18]).*
8. $(\tau, 1, s)$ $s\in[-1, 1]$ *then* ψ *is a s-convex extended function on* $[0,+\infty)$ *([29]).*
9. (τ, m, s) $s\in(0, 1]$ *then* ψ *is a (s, m)-convex extended function on* $[0,+\infty)$ *([26]).*
10. $(\tau^{a}, 1, s)$ *with* $a\in(0, 1]$, *then* ψ *is a (a, s)-convex function on* $[0,+\infty)$ *([6]).*
11. $(\tau^{a}, m, 1)$ *with* $a\in(0, 1]$, *then* ψ *is a (a, m)-convex function on* $[0,+\infty)$ *([21]).*
12. (τ^{a}, m, s) *with* $a\in(0, 1]$, *then* ψ *is a s-(a, m)-convex function on* $[0,+\infty)$ *([30]).*
13. $(h(\tau), m, 1)$ *then we have a variant of the* (h, m)*-convex function on* $[0,+\infty)$ *([25]).*
14.

The differential operators that we will use in our work are the following:

Definition 4. *Let* $\alpha>0$, *and* α /=1, 2, 3, , $n=[\alpha]+1$, $f\in AC^{n}[a, b]$, *the space of functions that have the n-th absolutely continuous derivatives. The weighted Caputo derivatives of the right-hand side and the left-hand side of order* α *are defined as follows:*

$$({}_{n}^{C}D_{v_1+}^{w'}f(v_2)=\int_{v_1}^{v_2}w'\left[\frac{v_2-x}{\frac{v_2-v_1}{r+1}}\right]f^{(n)}(x)dx,$$

$$({}_{n}^{C}D_{v_2-}^{w'}f(v_1)=\int_{v_1}^{v_2}w'\left[\frac{x-v_1}{\frac{v_2-v_1}{r+1}}\right]f^{(n)}(x)dx.$$

Remark 5. *Readers can verify that if in the previous definition we put* $r=0$ *and*

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$w'(z)=\dfrac{z^{\alpha-n+1}}{\Gamma(n-\alpha)(v_2-v_1)^{\alpha-n+1}}$, *we obtain the classic Caputo Derivative ([9]). Clearly the Caputo-Fabrizio Derivative ([10]) and the Atangan-Baleanu Derivative ([3]) can also be obtained without much difficulty.*

In this work we obtain different variants of the Hermite-Hadamard inequality, within the framework of *s*-convex functions and modified (*h*, *m*)-convex functions, using weighted Caputo derivatives.

2 Main results

Theorem 6. *Let f be a positive function such that $f\in C^n[a,b]$. If $f^{(n)}$ is s-convex function in the second sense, then we have the following inequality:*

$$f^{(n)}\left(\frac{a+b}{2}\right)\int_0^1 w(t)dt\leq\left(\frac{1}{2}\right)^s\frac{(r+1)}{(b-a)}\left[\left({}^{C}_{n}D^{w}_{\left(\frac{a+rb}{r+1}\right)+}f\right)(b)+\left({}^{C}_{n}D^{w}_{\left(\frac{ra+b}{r+1}\right)}-f\right)(a)\right]$$
$$\leq\left(\frac{1}{2}\right)^s\left(\left(f^{(n)}(a)+f^{(n)}(b)\right)\int_0^1 w(t)\left(\frac{t}{r+1}\right)^s dt+m\left(f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right)\int_0^1 w(t)\left(\frac{r+1-t}{r+1}\right)^s dt\right). \tag{3}$$

Proof. For $x,y\in[0,+\infty)$, $t=1/2$ and $m=1$, we have

$$f^{(n)}\left(\frac{x+y}{2}\right)=f^{(n)}\left(\frac{1}{2}x+\frac{1}{2}y\right)\leq\left(\frac{1}{2}\right)^s\left(f^{(n)}(x)+f^{(n)}(y)\right).$$

Making $x=\frac{t}{r+1}a+\left(\frac{r+1-t}{r+1}\right)b,\ y=\frac{t}{r+1}b+\left(\frac{r+1-t}{r+1}\right)a,$ with t∈[0; 1], we have

$$f^{(n)}\left(\frac{a+b}{2}\right)\leq\left(\frac{1}{2}\right)^s\left(f^{(n)}\left(\frac{t}{r+1}a+\left(\frac{r+1-t}{r+1}\right)b\right)+f^{(n)}\left(\frac{t}{r+1}b+\left(\frac{r+1-t}{r+1}\right)a\right)\right). \tag{4}$$

Multiplying both members of the previous inequality by $w(t)$, integrating with respect to the variable
t between 0 and 1, and changing the variables, we obtain the first inequality of (3).

$$f^{(n)}\int_0^1 w(t)dt\leq(\frac{1}{2})^s\int_0^1 w(t)f^{(n)}\left(\frac{t}{r+1}a+(\frac{r+1-t}{r+1})b\right)dt+(\frac{1}{2})^s\int_0^1 w(t)f^{(n)}\left(\frac{t}{r+1}b+(\frac{r+1-t}{r+1})a\right)dt,$$

$$f^{(n)}\left(\frac{a+b}{2}\right)\int_0^1 w(t)dt\leq(\frac{1}{2})^s\frac{(r+1)}{(a-b)}\int_b^{\frac{a+rb}{r+1}} w\left[\frac{\frac{(x-b)}{a-b}}{(r+1)}\right]f^{(n)}(x)dx+(\frac{1}{2})^s\frac{(r+1)}{(b-a)}\int_a^{\frac{ra+b}{r+1}} w\left[\frac{\frac{(x-b)}{a-b}}{(r+1)}\right]f^{(n)}(x)dx$$

$$f^{(n)}\left(\frac{a+b}{2}\right)\int_0^1 w(t)dt\leq(\frac{1}{2})^s\frac{(r+1)}{(b-a)}\int_{\frac{a+rb}{r+1}}^{b} w\left[\frac{\frac{(b-x)}{(b-a)}}{(r+1)}\right]f^{(n)}(x)dx+(\frac{1}{2})^s\frac{(r+1)}{(b-a)}\int_a^{\frac{ra+b}{r+1}} w\left[\frac{\frac{(b-x)}{(b-a)}}{(r+1)}\right]f^{(n)}(x)dx$$

$$f^{(n)}\left(\frac{a+b}{2}\right)\int_0^1 w(t)dt\leq(\frac{1}{2})^s\frac{(r+1)}{(b-a)}\left({}^{C}_{n}D^{w}_{(\frac{ra+b}{r+1})}-f\right)(b)+(\frac{1}{2})^s\frac{(r+1)}{(b-a)}\left({}^{C}_{n}D^{w}_{(\frac{ra+b}{r+1})}-f\right)(a).$$

From the right side of (4), we obtain

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$$(\frac{1}{2})^s\left[f^{(n)}\left(\frac{t}{r+1}a+(\frac{r+1-t}{r+1})b\right)+f^{(n)}\left(\frac{t}{r+1}b+(\frac{r+1-t}{r+1})a\right)\right]$$

$$=(\frac{1}{2})^s\left[f^{(n)}\left(\frac{t}{r+1}a+m(\frac{r+1-t}{r+1})\frac{b}{m}\right)+f^{(n)}\left(\frac{t}{r+1}b+m(\frac{r+1-t}{r+1})\frac{a}{m}\right)\right]$$

$$\le(\frac{1}{2})^s\left[(\frac{t}{r+1})^s f^{(n)}(a)+\left(\frac{r+1-t}{r+1}\right)^s mf^{(n)}(\frac{b}{m})\right]$$

$$+(\frac{1}{2})^s\left[(\frac{t}{r+1})^s f^{(n)}(b)+\left(\frac{r+1-t}{r+1}\right)^s mf^{(n)}(\frac{a}{m})\right]$$

$$=(\frac{1}{2})^s\left[\left(\frac{t}{r+1}\right)^s(f^{(n)}(a)+f^{(n)}(b))+\left(\frac{r+1-t}{r+1}\right)^s\left(mf^{(n)}(\frac{a}{m})+mf^{(n)}(\frac{b}{m})\right)\right].$$

Multiplying by $w(t)$, integrating with respect to the variable t between 0 and 1, we obtain the right side of (3). In this way, the proof is complete. ■

Theorem 7. *Let f be a positive function such that $f\in C^n$ $[a, b]$. If $f^{(n)}$ is a modified* (h, m)*-convex function of the second type with $m\in(0, 1]$ and $0<v_1<mv_2<+\infty$, then we have the following*
inequality:

$$f^{(n)}(\frac{a+b}{2})\int_0^1 w(t)dth^s(\frac{1}{2})\frac{(r+1)}{(b-a)}({}_n^C D^w_{(\frac{a+rb}{r+1})+} f)(b)+(1-h(\frac{1}{2}))^s\frac{(r+1)}{(b-a)}({}_n^C D^w_{(\frac{ra+b}{r+1})-} f)(a)$$

$$(h^s(\frac{1}{2})f^{(n)}(a)+(1-h(\frac{1}{2}))^s f^{(n)}(b))\int_0^1 w(t)h^s(\frac{t}{r+1})dt \tag{5}$$

$$+m(h^s(\frac{1}{2})f^{(n)}(\frac{b}{m})+(1-h(\frac{1}{2}))^s f^{(n)}(\frac{a}{m}))\int_0^1 w(t)(1-h(\frac{r+1-t}{r+1}))^s dt.$$

Proof. For $x, y\in[0,+\infty)$, $t= 1/2$ and $m=1$, we have

$$f^{(n)}(\frac{x+y}{2})\le h^s(\frac{1}{2})f^{(n)}(x)+(1-h(\frac{1}{2}))^s f^{(n)}(y).$$

Making $x=\frac{t}{r+1}a+(\frac{r+1-t}{r+1})b, y=\frac{t}{r+1}b+(\frac{r+1-t}{r+1})a,$ with $t\in[0, 1]$, we have

$$f^{(n)}(\frac{a+b}{2})\le h^s(\frac{1}{2})f^{(n)}(\frac{t}{r+1}a+(\frac{r+1-t}{r+1})b)+(1-h(\frac{1}{2}))^s f^{(n)}(\frac{t}{r+1}b+(\frac{r+1-t}{r+1})a). \tag{6}$$

Multiplying both members of the previous inequality by $w(t)$, integrating with respect to the variable
t between 0 and 1, and changing the variables, we obtain the first inequality of (5).

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$$f^{(n)}(\frac{a+b}{2})\int_0^1 w(t)dt \le h^s(\frac{1}{2})\int_0^1 w(t)\,f^{(n)}\left(\frac{t}{r+1}a+\left(\frac{r+1-t}{r+1}\right)b\right)dt$$

$$+(1-h(\frac{1}{2}))^s\int_0^1 w(t)\,f^{(n)}\left(\frac{t}{r+1}b+\left(\frac{r+1-t}{r+1}\right)a\right)dt,$$

$$f^{(n)}(\frac{a+b}{2})\int_0^1 w(t)dt \le h^s(\frac{1}{2})\frac{(r+1)}{(a-b)}\int_b^{\frac{a+rb}{r+1}} w\left[\frac{(x-a)}{\frac{(b-a)}{(r+1)}}\right]f^{(n)}(x)dx$$

$$+(1-h(\frac{1}{2}))^s\frac{(r+1)}{(b-a)}\int_a^{\frac{ra+b}{r+1}} w\left[\frac{(x-a)}{\frac{(b-a)}{(r+1)}}\right]f^{(n)}(x)dx,$$

$$f^{(n)}(\frac{a+b}{2})\int_0^1 w(t)dt \le h^s(\frac{1}{2})\frac{(r+1)}{(b-a)}\int_{\frac{a+rb}{r+1}}^{b} w\left[\frac{(b-x)}{\frac{(b-a)}{(r+1)}}\right]f^{(n)}(x)dx$$

$$+(1-h(\frac{1}{2}))^s\frac{(r+1)}{(b-a)}\int_a^{\frac{ra+b}{r+1}} w\left[\frac{(b-x)}{\frac{(b-a)}{(r+1)}}\right]f^{(n)}(x)dx,$$

$$f^{(n)}(\frac{a+b}{2})\int_0^1 w(t)dt \le h^s(\frac{1}{2})\frac{(r+1)}{(b-a)}({}_n^C D^w_{(\frac{a+rb}{r+1})+}f)(b)$$

$$+(1-h(\frac{1}{2}))^s\frac{(r+1)}{(b-a)}({}_n^C D^w_{(\frac{ra+b}{r+1})-}f)(a).$$

From the right side of (6), we obtain

$$h^s(\frac{1}{2})f^{(n)}\left[\frac{t}{r+1}a+(\frac{r+1-t}{r+1})b\right]+(1-h(\frac{1}{2}))^s f^{(n)}\left[\frac{t}{r+1}b+(\frac{r+1-t}{r+1})a\right]$$

$$=h^s(\frac{1}{2})f^{(n)}\left[\frac{t}{r+1}a+m(\frac{r+1-t}{r+1})\frac{b}{m}\right]+(1-h(\frac{1}{2}))^s f^{(n)}\left[\frac{t}{r+1}b+m(\frac{r+1-t}{r+1})\frac{a}{m}\right]$$

$$\le h^s(\frac{1}{2})\left[f^{(n)}(a)h^s\left(\frac{t}{r+1}\right)+mf^{(n)}(\frac{b}{m})\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s\right]$$

$$+(1-h(\frac{1}{2}))^s\left[f^{(n)}(b)h^s\left(\frac{t}{r+1}\right)+mf^{(n)}(\frac{a}{m})\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s\right].$$

Multiplying by $w(t)$, integrating with respect to the variable t between 0 and 1, we obtain the right side of (5). In this way, the proof is complete. ■

Remark 8. *With* $w(t)=t^{n-\frac{\alpha}{k}}$, *if in the previous Theorem we make* $r=0$ *and consider functions* (h, m)*-convex, that is,* $s=1$, *we obtain Theorem 2.1 of [22]. If we put* $r=1$, *then they are particular cases of the previous Theorem, Theorem 6 of [12], Theorem 4 of [13] with* $k=1$ *and Theorem 2.2 of [19], all for convex functions, that is,* $h(t)=t$, $m=s=1$.

The following lemma will be useful to prove future theorems.

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Lemma 9. *Let* f *be a real function defined on the real interval* $[a, b]$ *and differentiable on* (a, b). *If* $f' \in L_1(a, b)$, *and* $w(t)$ *is a function differentiable on* (a, b), *then we have the following equality:*

$$\left\{-w(1)(f^{(n)}(\frac{a+rb}{r+1})+f^{(n)}(\frac{ra+b}{r+1}))+w(0)(f^{(n)}(a)+f^{(n)}(b))\right\} \quad (7)$$

$$+\frac{r+1}{b-a}\left[({}^{C}_{n}D^{w'}_{(\frac{ra+b}{r+1})}-f(a)+{}^{C}_{n}D^{w'}_{(\frac{a+rb}{r+1})}+f(b)\right]$$

$$=\frac{b-a}{r+1}\int_0^1 w(t)\left[f^{(n+1)}\left(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\right)-f^{(n+1)}\left(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\right)\right]dt.$$

Proof. First of all let us note that,

$$\int_0^1 w(t)\left[f^{(n+1)}\left(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\right)-f^{(n+1)}\left(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\right)\right]dt$$

$$=\int_0^1 w(t)f^{(n+1)}\left(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\right)dt-\int_0^1 w(t)f^{(n+1)}\left(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\right)dt$$

$$=M_1-M_2.$$

Integrating by parts, we have

$$M_1=\frac{r+1}{b-a}\left(-w(1)f^{(n)}(\frac{a+rb}{r+1})+w(0)f^{(n)}(b)\right)+\frac{(r+1)^2}{(b-a)^2}\int_a^{\frac{ra+b}{r+1}} w'\left(\frac{\frac{x-a}{b-a}}{r+1}\right)f^{(n)}(x)dx,$$

because

$$\int_0^1 w'(t)f^{(n)}(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b)dt=\frac{n+1}{b-a}\int_a^{\frac{ra+b}{r+1}} w'\left(\frac{\frac{x-a}{b-a}}{r+1}\right)f^{(n)}(x)dx.$$

In an analogous way,

$$M_2=\frac{r+1}{b-a}\left(w(1)f^{(n)}(\frac{ra+b}{r+1})-w(0)f^{(n)}(a)\right)-\frac{(r+1)^2}{(b-a)^2}\int_{\frac{a+rb}{r+1}}^{b} w'\left(\frac{\frac{b-x}{b-a}}{r+1}\right)f^{(n)}(x)dx.$$

By making M_1-M_2, we obtain the desired equality. ■

Using Lemma 9, we obtain the following results.

Theorem 10. *Let* f *be a positive real function defined on* $[a, b]\subset \mathrm{R}$, *such that* $f^{(n)}\in L_1(a, mb)$. *If* $|f^{(n)}|$ *is s-convex in* $\left[a, \frac{b}{m}\right]$, *we have the following inequality:*

$$\mathrm{S}(a,b,f,w)\le\frac{b-a}{r+1}(\left|f^{(n+1)}(a)\right|+\left|f^{(n+1)}(b)\right|)\mathrm{T}+m(\left|f^{(n+1)}(\frac{a}{m})\right|+\left|f^{(n+1)}(\frac{b}{m})\right|)\mathrm{F}. \quad (8)$$

where $\mathrm{S}(a,b,f,w)$ *is the absolute value of the left side of (7),*

$$\mathrm{T}=\int_0^1|w(t)|(\frac{t}{r+1})^s dt,$$

$$\mathrm{F}=\int_0^1|w(t)|(\frac{r+1-t}{r+1})^s dt.$$

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Proof. From Lemma 9, we have

$$\left|\int_0^1 w(t)\Big(f^{(n+1)}\Big(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\Big)-f^{(n+1)}\Big(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\Big)\Big)dt\right|$$

$$\leq \int_0^1 |w(t)|\left|f^{(n+1)}\Big(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\Big)\right|dt+\int_0^1 |w(t)|\left|f^{(n+1)}\Big(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\Big)\right|dt.$$

Using the s-convexity of $|f^{(n+1)}|$, we have

$$\int_0^1 |w(t)|\left|f^{(n+1)}\Big(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\Big)\right|dt\int_0^1 |w(t)|\Big(\Big(\frac{t}{r+1}\Big)^s\left|f^{(n+1)}(a)\right|+m\Big(\frac{r+1-t}{r+1}\Big)^s\left|f^{(n+1)}\Big(\frac{b}{m}\Big)\right|\Big)dt$$

$$=\left|f^{(n+1)}(a)\right|\int_0^1 |w(t)|\Big(\frac{t}{r+1}\Big)^s dt+m\left|f^{(n+1)}\Big(\frac{b}{m}\Big)\right|\int_0^1 |w(t)|\Big(\frac{r+1-t}{r+1}\Big)^s dt. \quad (9)$$

Analogously,

$$\int_0^1 |w(t)|\left|f^{(n+1)}\Big(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\Big)\right|dt$$

$$\leq\left|f^{(n+1)}(b)\right|\int_0^1 |w(t)|\Big(\frac{t}{r+1}\Big)^s dt+m\left|f^{(n+1)}\Big(\frac{a}{m}\Big)\right|\int_0^1 |w(t)|\Big(\frac{r+1-t}{r+1}\Big)^s dt. \quad (10)$$

From (9) and (10), we obtain (8).

Theorem 11. *Let f be a positive real function defined on $[a, b]\subset \mathrm{R}$, such that $f^{(n)}\in L_1(a, mb)$. If $|f^{(n)}|$ es (h, m)-modified convex of the second type in $\left[a,\frac{b}{m}\right]$, we have the following inequality:*

$$\mathrm{I}(a,b,f,w)\leq\frac{b-a}{r+1}\Big(\left|f^{(n+1)}(a)\right|+\left|f^{(n+1)}(b)\right|\Big)\mathrm{G}+m\Big(\left|f^{(n+1)}\Big(\frac{a}{m}\Big)\right|+\left|f^{(n+1)}\Big(\frac{b}{m}\Big)\right|\Big)\mathrm{H}. \quad (11)$$

where $\mathrm{I}(a,b,f,w)$ *is the absolute value of the left side of (7),*

$$\mathrm{G}=\int_0^1 |w(t)|\,h^s\Big(\frac{t}{r+1}\Big)dt,$$

$$\mathrm{H}=\int_0^1 |w(t)|\Big(1-h\Big(\frac{r+1-t}{r+1}\Big)\Big)^s dt.$$

Proof. From Lemma 9, we have

$$\left|\int_0^1 w(t)\Big(f^{(n+1)}\Big(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\Big)-f^{(n+1)}\Big(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\Big)\Big)dt\right|$$

$$\leq \int_0^1 |w(t)|\left|f^{(n+1)}\Big(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\Big)\right|dt+\int_0^1 |w(t)|\left|f^{(n+1)}\Big(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\Big)\right|dt.$$

Using the modified (h;m)-convexity of $|f^{(n+1)}|$, we have

$$\int_0^1 |w(t)|\left|f^{(n+1)}\Big(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\Big)\right|dt\int_0^1 |w(t)|\Big(h^s\Big(\frac{t}{r+1}\Big)\left|f^{(n+1)}(a)\right|+m\Big(1-h\Big(\frac{r+1-t}{r+1}\Big)\Big)^s\left|f^{(n+1)}\Big(\frac{b}{m}\Big)\right|\Big)dt$$

$$=\left|f^{(n+1)}(a)\right|\int_0^1 |w(t)|h^s\Big(\frac{t}{r+1}\Big)dt+m\left|f^{(n+1)}\Big(\frac{b}{m}\Big)\right|\int_0^1 |w(t)|\Big(1-h\Big(\frac{r+1-t}{r+1}\Big)\Big)^s dt. \quad (12)$$

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Analogously,

$$\int_0^1 |w(t)| \left| f^{(n+1)}\left(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\right)\right| dt$$
$$\leq \left| f^{(n+1)}(b)\right| \int_0^1 |w(t)| h^s\left(\frac{t}{r+1}\right)dt + m\left| f^{(n+1)}\left(\frac{a}{m}\right)\right| \int_0^1 |w(t)| \left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt. \tag{13}$$

From (12) and (13), we obtain (11).■

Putting $w(t)=t^{n-\frac{\alpha}{k}}$ we have the following remark.**Remark 12.** *Considering $r=0$, it is easy to check that the first part of Theorem 2.7 of [22] for functions (h, m)-convex; with $r=1$ the Theorem 7 (with $q=1$) of [12] for convex functions and Theorem 5 of [13], Theorem 3.2 of [19] (the case $q=1$ of both for convex functions), they are all particular cases of the previous theorem.*

The above result can be improved if we impose additional conditions on $|f^{(n+1)}|^q$.

Theorem 13. *Let f be a positive real function defined on $[v_1, v_2]\subset \mathbb{R}$, such that $f^{(n+1)}\in L_1(v_1, mv_2)$. If $|f^{(n+1)}|^q$ is a s-convex function of the second sense on $\left[v_1, \frac{v_2}{m}\right]$, we have the following inequality;*

$$\mathrm{S}(a,b,f,w) \leq \frac{b-a}{r+1}\mathrm{U}_p \sum_{i=1}^{2}\left[\left| f^{(n+1)}(v_{3-i})\right| \mathrm{W} + m\left| f^{(n+1)}\left(\frac{v_i}{m}\right)\right| \mathrm{X}\right]^{\frac{1}{q}}, \tag{14}$$

where $\mathrm{U}_p = \left(\int_0^1 |w(t)|^p\, dt\right)^{\frac{1}{p}}, \mathrm{W} = \int_0^1 \left(\frac{t}{r+1}\right)^s dt, \mathrm{L} = \int_0^1 \left(\frac{r+1-t}{r+1}\right)^s dt.$

Proof. As in some previous Theorem, we have

$$\left|\int_0^1 w(t)\left(f^{(n+1)}\left(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\right) - f^{(n+1)}\left(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\right)\right)dt\right|$$
$$\leq \int_0^1 |w(t)| \left| f^{(n+1)}\left(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\right)\right| dt + \int_0^1 |w(t)| \left| f^{(n+1)}\left(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\right)\right| dt$$
$$= N_1 + N_2.$$

From Hölder's inequality, we have

$$N_1 \leq \left(\int_0^1 |w(t)|^p\, dt\right)^{\frac{1}{p}} \left(\int_0^1 \left| f^{(n+1)}\left(\frac{t}{r+1}v_1+\frac{r+1-t}{r+1}v_2\right)\right|^q dt\right)^{\frac{1}{q}}, \text{(15)and}$$

$$N_2 \leq \left(\int_0^1 |w(t)|^p\, dt\right)^{\frac{1}{p}} \left(\int_0^1 \left| f^{(n+1)}\left(\frac{t}{r+1}v_2+\frac{r+1-t}{r+1}v_1\right)\right|^q dt\right)^{\frac{1}{q}}, \tag{16}$$

for $\frac{1}{p}+\frac{1}{q}=1$. Using the *s*-convexity of the second sense of $|f^{(n+1)}|^q$, we obtain from (15) and (16);

$$\int_0^1\left|f^{(n+1)}\left(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\right)\right|^q dt$$
$$\le\left|f^{(n+1)}(a)\right|^q\int_0^1\left(\frac{t}{r+1}\right)^s dt+m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q\int_0^1\left(\frac{r+1-t}{r+1}\right)^s dt, \qquad (17)$$

and

$$\int_0^1\left|f^{(n+1)}\left(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\right)\right|^q dt$$
$$\le\left|f^{(n+1)}(b)\right|^q\int_0^1\left(\frac{t}{r+1}\right)^s dt+m\left|f^{(n+1)}\left(\frac{a}{m}\right)\right|^q\int_0^1\left(\frac{r+1-t}{r+1}\right)^s dt. \qquad (18)$$

Substituting (17), (18) into (15) and (16), we obtain the desired inequality.■

Theorem 14 *Let* f *be a positive real function defined on* $[a,b]\subset\mathrm{R}$, *such that* $f^{(n+1)}\in L_1(a,mb)$. *If* $|f^{(n+1)}|^q$ *is a* (h, m)*-convex function of the second type on* $\left[a,\frac{b}{m}\right]$, *we have the following inequality;*

$$\mathrm{I}(a,b,f,w)\le\frac{b-a}{r+1}\mathrm{J}_p\sum_{i=1}^{2}\left[\left|f^{(n+1)}(v_{3-i})\right|\mathrm{K}+m\left|f^{(n+1)}\left(\frac{v_i}{m}\right)\right|\mathrm{L}\right]^{\frac{1}{q}}, (19)$$

where

$$\mathrm{J}_p=\left(\int_0^1|w(t)|^p\,dt\right)^{\frac{1}{p}},\mathrm{K}=\int_0^1 h^s\left(\frac{t}{r+1}\right)dt,\mathrm{L}=\int_0^1\left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt.$$

Proof. As in some previous Theorem, we have

$$\left|\int_0^1 w(t)\left(f^{(n+1)}\left(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\right)-f^{(n+1)}\left(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\right)\right)dt\right|$$
$$\le\int_0^1|w(t)|\left|f^{(n+1)}\left(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\right)\right|dt+\int_0^1|w(t)|\left|f^{(n+1)}\left(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\right)\right|dt.$$

From Hölder’s inequality, we have

$$\int_0^1|w(t)|\left|f^{(n+1)}\left(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\right)\right|dt$$
$$\le\left(\int_0^1|w(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n+1)}\left(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\right)\right|^q dt\right)^{\frac{1}{q}} \qquad (20)$$
$$=\left(\int_0^1|w(t)|^p dt\right)^{\frac{1}{p}}R_1.$$

and

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$$\int_0^1 |w(t)| \left| f^{(n+1)}(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a)\right| dt$$

$$\le \left(\int_0^1 |w(t)|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left| f^{(n+1)}(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a)\right|^q dt\right)^{\frac{1}{q}} \quad (21)$$

$$= \left(\int_0^1 |w(t)|^p dt\right)^{\frac{1}{p}} R_2.$$

for $\frac{1}{p}+\frac{1}{q}=1.$ Using the s-convexity of the second sense of $|f^{(n+1)}|^q$, we obtain from (20) and (21);

$$R_1 \le \left| f^{(n+1)}(a)\right|^q \int_0^1 h^s\left(\frac{t}{r+1}\right)dt + m\left| f^{(n+1)}(\frac{b}{m})\right|^q \int_0^1 \left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt, \quad (22)$$

and

$$R_2 \le \left| f^{(n+1)}(b)\right|^q \int_0^1 h^s\left(\frac{t}{r+1}\right)dt + m\left| f^{(n+1)}(\frac{a}{m})\right|^q \int_0^1 \left(1-h\left(\frac{r+1-t}{r+1}\right)\right)^s dt. \quad (23)$$

Substituting (22), (23) into (20) and (21), we obtain the desired inequality.■

As before, if put $w(t)=t^{n-\frac{\alpha}{k}}$ we have:

Remark 15. *Theorem 8 (*$q>1$*) of [12], Theorem 6 of [13], the second part of Theorem 2.7 of [22] and Theorem 3.2 (*$q>1$*) of [19] can be derived from the previous result, for different values of* r *and different notions of convexity.*

Theorem 16. *Let* f *be a positive real function defined on* $[a,b]\subset$□, *such that* $f^{(n+1)}\in L_1(a,mb)$. *If* $|f^{(n+1)}|^q$, $q>1$, *is a s-convex function of the second sense in* $\left[a,\frac{b}{m}\right]$ *we have the following inequality;*

$$\mathrm{S}(a,b,f,w) \le \frac{b-a}{r+1}\mathrm{U}_q \sum_{i=1}^{2}\left(| f^{(n+1)}(v_{3-i})|^q\, \mathrm{T} + m\, | f^{(n+1)}(\frac{v_i}{m})|^q\, \mathrm{F}\right)^{\frac{1}{q}} \quad (24)$$

where

$$\mathrm{U}_q = \left(\int_0^1 |w(t)|\, dt\right)^{1-\frac{1}{q}}.$$

Proof. We know that

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$$\left|\int_0^1 w(t)(f^{(n+1)}(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b)-f^{(n+1)}(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a))dt\right|$$

$$\leq \int_0^1 |w(t)|\left|f^{(n+1)}(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b)\right|dt$$

$$+\int_0^1 |w(t)|\left|f^{(n+1)}(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a)\right|dt$$

$$=Z_1+Z_2.$$

Using the mean power inequality, we obtain

$$Z_1 \leq \left(\int_0^1 |w(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|f^{(n+1)}(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b)\right|^q dt\right)^{\frac{1}{q}}, \quad (25)$$

and

$$Z_2 \leq \left(\int_0^1 |w(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|f^{(n+1)}(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a)\right|^q dt\right)^{\frac{1}{q}}. \quad (26)$$

Using the modified s-convexity of $|f^{(n+1)}|^q$, we have

$$\int_0^1 \left|f^{(n+1)}(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b)\right|^q dt \leq |f^{(n+1)}(a)|^q\,\mathrm{T}+m\left|f^{(n+1)}(\frac{b}{m})\right|^q \mathrm{F}, \quad (27)$$

and

$$\int_0^1 \left|f^{(n+1)}(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a)\right|^q dt \leq |f^{(n+1)}(b)|^q\,\mathrm{T}+m\left|f^{(n+1)}(\frac{a}{m})\right|^q \mathrm{F}. \quad (28)$$

Substituting (27) and (28) into (25) and (26) respectively, we arrive at (24).■

Theorem 17. *Let f be a positive real function defined on $[a, b]\subset \mathrm{R}$, such that $f^{(n+1)}\in L_1(a, mb)$. If $|f^{(n+1)}|^q$, $q>1$, is a modified (h, m)-convex function of the second type in $\left[a,\frac{b}{m}\right]$, we have the following inequality;*

$$\mathrm{I}(v_1,v_2,f,w)\leq \frac{v_2-v_1}{r+1}\mathrm{J}_q\sum_{i=1}^{2}\left(\left(|f^{(n+1)}(v_{3-i})|^q\,\mathrm{G}+m|f^{(n+1)}(\frac{v_i}{m})|^q\,\mathrm{H}\right)^{\frac{1}{q}}\right), \quad (29)$$

where

$$\mathrm{J}_q=\left(\int_0^1 |w(t)|\,dt\right)^{1-\frac{1}{q}}.$$

Proof. We know that

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$$\left|\int_0^1 w(t)(f^{(n+1)}(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b)-f^{(n+1)}(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a))dt\right|$$

$$\leq\int_0^1|w(t)|\left|f^{(n+1)}(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b)\right|dt$$

$$+\int_0^1|w(t)|\left|f^{(n+1)}(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a)\right|dt$$

$$=Z_1+Z_2.$$

Using the mean power inequality, we obtain

$$Z_1\leq\left(\int_0^1|w(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f^{(n+1)}(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b)\right|^q dt\right)^{\frac{1}{q}},$$

and

$$Z_2\leq\left(\int_0^1|w(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f^{(n+1)}(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a)\right|^q dt\right)^{\frac{1}{q}}.$$

Using the modified (h,m)-convexity of $|f^{(n+1)}|^q$, we have

$$Z_1\leq\left(\int_0^1|w(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f^{(n+1)}(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b)\right|^q dt\right)^{\frac{1}{q}} \quad (30)$$

$$\leq\left(\int_0^1|w(t)|\,dt\right)^{1-\frac{1}{q}}(\left|f^{(n+1)}(a)\right|^q \mathrm{G}+m\left|f^{(n+1)}(\frac{b}{m})\right|^q \mathrm{H})^{\frac{1}{q}},$$

and

$$Z_2\leq\left(\int_0^1|w(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f^{(n+1)}(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a)\right|^q dt\right)^{\frac{1}{q}} \quad (31)$$

$$\leq\left(\int_0^1|w(t)|\,dt\right)^{1-\frac{1}{q}}(\left|f^{(n+1)}(b)\right|^q \mathrm{G}+m\left|f^{(n+1)}(\frac{a}{m})\right|^q \mathrm{H})^{\frac{1}{q}}.$$

From (30) and (31), we arrive at (29). ■

Remark 18. *Under above assumption on $w(t)$, this last result covers Theorem 7 ($q>1$) of [12], Theorem 5 from [13] ($q>1$) and Theorem 3.3 from [19], for different values of r and different definitions of convexity.*

Remark 19. *All previous results contain many of those reported in the literature, particularizing the function w as well as for different notions of convexity.*

3 Conclusions

In this work, we have derived new versions of the well-known Hermite-Hadamard Inequality in the framework of the weighted fractional derivative of Caputo, the generality of the results has been demonstrated by showing that known results from the literature are particular cases of ours.

References

[1] M. A. Ali, J. E. Nápoles V., A. Kashuri, Z. Zhang, Fractional non conformable Hermite-Hadamard inequalities for generalized-convex functions, Fasciculi Mathematici, Nr 64 2020, 5-16 http://dx.doi.org/10.21008/j.0044-4413.2020.0007.

[2] M. E. Amlashi, M. Hassani, More on the Hermite-Hadamard inequality, Int. J. Nonlinear Anal.Appl. 12 (2021) No. 2, 2153-2159. http://dx.doi.org/10.22075/ijnaa.2021.21753.2293

[3] A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel:theory and application to Heat transfer model. Therm. Sci. 20, 763-769 (2016).

[4] B. Bayraktar, J. E. Nápoles V., A note on Hermite-Hadamard integral inequality for (h, m)-convex modified functions in a generalized framework, submited.

[5] S. Bermudo, P. Kórus, J. E. Nápoles V., On q-Hermite-Hadamard inequalities for general convex functions, Acta Math. Hungar. 162, 364-374 (2020) https://doi.org/10.1007/s10474-020-01025-6

[6] M. Bilal, M. Imtiaz, A. R. Khan, I. U. Khan, M. Zafran, Generalized Hermite-Hadamard inequal-ities for s-convex functions in the mixed kind, submitted

[7] W. W. Breckner, Stetigkeitsaussagen fur eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen, Pupl. Inst. Math., 23 (1978), 13-20.

[8] A. M. Bruckner, E. Ostrow, Some function classes related to the class of convex functions, PacificJ. Math. 12 (1962), 1203-1215.

[9] M. Caputo, Elasticit e Dissipazione, Bologna: Zanichelli, 1969.

[10] M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, Prog.Fract. Differ. Appl. 1 (2) (2015) 7385

[11] S. S. Dragomir, J. Pecaric, L. E. Persson, Some inequalities of Hadamard type, Soochow J. Math., 21 (1995), 335-241.

[12] G. Farid, A. Javed, A. U. Rehman, M. I. Qureshi, On Hadamard-type inequalities for differ-entiable functions via Caputo k-fractional derivatives, Cogent Mathematics (2017), 4: 1355429 https://doi.org/10.1080/23311835.2017.1355429

[13] G. Farid, S. Naqvi, A. U. Rehman, A version of the Hadamard inequality for Caputo fractional derivatives and related results, RGMIA Research Report Collection, 2017, 11 pp, 20, Article 59.

[14] A. E. Farissi, Simple proof and refinement of Hermite-Hadamard inequality, Journal of Mathe-matical Inequalities, Volume 4, Number 3 (2010), 365-369

[15] P. M. Guzmán, J. E. Nápoles V., Y. Gasimov, Integral inequalities within the framework of generalized fractional integrals, Fractional Differential Calculus, Volume 11, Number 1 (2021), 69-84 doi:10.7153/fdc-2021-11-05

[16] J. Hadamard, Étude sur les propri´et´es des fonctions enti´eres et en particulier d’une fonction consider´ee par Riemann, J. Math. Pures A 9, 171-216 (1893).

[17] C. Hermite, Sur deux limites d’une int´egrale d´efinie, Mathesis3, 82 (1883).

[18] H. Hudzik, L. Maligranda, Some remarks on s-convex functions, Aequationes Math. 48 (1994), 1, 100-111

[19] S. M. Kang, G. Farid, W. Nazeer, S. Naqvi, A version of the Hadamard inequality for Caputo fractional derivatives and related results. Journal of Computational Analysis & Applications, 27(6), 2019, 962-972.

[20] D. S. Marinescu, M. Monea, A Very Short Proof of the HermiteHadamard Inequalities, The American Mathematical Monthly, 127:9, 2020, 850-851

Received: 16-01-2024 Revised: 12-02-2024 Accepted: 07-03-2024

[21] V. G. Mihesan, A generalization of the convexity, Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca (Romania) (1993).

[22] L. N. Mishra, Q. U. Ain, G. Farid, A. U. Rehman, k-fractional integral inequalities for (h,m)-convex functions via Caputo k-fractional derivatives, Korean J. Math. 27 (2019), 2, 357-374 https://doi.org/10.11568/kjm.2019.27.2.357

[23] J. E. Nápoles Vald´es, F. Rabossi, A. D. Samaniego, Convex functions: Ariadne's thread or Char-lotte's spiderweb, Advanced Mathematical Models & Applications Vol.5, No.2, 2020, pp.176-191

[24] J. E. Nápoles Vald´es, J. M. Rodr´ıguez, J. M. Sigarreta, On Hermite-Hadamard type inequalities for non-conformable integral operators, *Symmetry* **2019**, *11*, 1108.

[25] M. E. Ö zdemir, A. O. Akdemri, E. Set, On (h, m)-convexity and Hadamard-type inequalities, Transylv. J. Math. Mech. 8(1), 51-58 (2016)

[26] J. Park, Generalization of Ostrowski-type inequalities for differentiable real (s, m)-convex map-pings, Far East J. Math. Sci., 49 (2011), 157-171. https://doi.org/10.1155/2020/1626091

[27] S. Simic, Some refinements of Hermite-Hadamard inequality and an open problem, Kragujevac Journal of Mathematics. 42(3) (2018), 349-356.

[28] G. Toader, Some generalizations of the convexity, Proceedings of the Colloquium on Approxima-tion and Optimization, University Cluj-Napoca, 1985, 329-338.

[29] B. Y. Xi, F. Qi, Inequalities of Hermite-Hadamard type for extended s-convex functions and applications to means, J. Nonlinear Convex. Anal. 16 (2015), 5, 873-890.

[30] B. Y. Xi, D.D. Gao, F. Qi, Integral inequalities of Hermite-Hadamard type for (α, s)-convex and (α, s, m)-convex functions, Italian Journal of Pure and Applied Mathematics N.44-2020 (499-510)

[31] R.Dhavaseelan, S. Jafari, M.R. Farahani, S. Broumi. On single-valued co-neutrosophic graphs. Neutrosophic Sets and Systems. 22, 2018, 180-187.

[32] M.R. Farahani, S. Jafari, S.A. Mohiuddine, M.t Cancan. Intuitionistic Fuzzy Stability of Generalized Additive Set-Valued Functional Equation via Fixed Point Method. Mathematical Statistician and Engineering Applications, 71(3s3), 142-154. (2022). https://www.philstat.org.ph/index.php/MSEA/article/view/355

[33] M.R. Farahani, J.E. Nápoles Valdes, M. Cancan, S. Jafari, C. Park. Fuzzy stability in the sense of Hyers-Ulam-Rassias of a functional equation on quadratic forms. Journal of Interdisciplinary Mathematics. 26(6), 1355-1365.https://doi.org/10.47974/JIM-1633

[34] S. Jafari, M.R. Farahani, M. Vanishree, N. Rajesh. Double fuzzy rarely continuous functions. Annals of Fuzzy Mathematics and Informatics. 26(1), 35-45. (2023).