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Computation of Lucky Number of Comb Graphs Cf_w, Cg_w, Ch_w and Triangular Snake, Alternate Triangular Snake Graphs

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Abstract
Let $\lambda: V(\Omega) \rightarrow \{1, 2, \ldots, z\}$ be a mapping of vertices of a graph Ω. Let $S(x)$ denote the sum of labels of the neighbors of the vertex x in Ω. If vertex x has degree zero, we put $S(x)=0$. A mapping λ is categorized as lucky labeling if $S(x)=S(y)$ for every pair of adjacent vertices x and y. The lucky number of graph Ω, denoted by η (Ω), is the least positive integer z used to label vertices to form lucky labeling. In this paper, we demonstrate that different families of comb graph and snake graph are lucky labeled graphs. We also calculate the exact value of the lucky number forthe aforementioned graphs.

Keywords: *Comb graphs, lucky labeling, lucky number, lucky labeled graph, snake graphs, triangular snake, alternate triangular snake.*
Mathematics Subject Classification:05C78.

1 Introduction
Graph labeling is a fascinating area of mathematics that has several real-life applications across various domains. Labeled graphs are used to model scenarios in game theory, help-ing to analyze strategic interactions and decision making processes. This concept finds applications in environmental modeling to study ecosystems, pollution spreading, environmental impact assessments. Additionally, it plays a significant role in various ma-chine leaning tasks such as graph-based semi-supervised learning, clustering, pattern recognition. Among the types of labeling, one particular type is lucky labeling, which we discuss in this paper. Lucky labeling concepts can be related to the efficient routing of data in communication networks. Lucky numbers, a concept derived from graph theory, have various applications in different contexts within the field. Lucky numbers are used in design and optimization of communication networks, transportation systems. This con-cept is helpful in finding efficient routes, minimizing congestion, improving overall network performance. Lucky numbers can be part of optimization

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algorithms that aim to find solutions to combinatorial optimization problems in graph theory, such as finding the shortest paths or maximum flows in networks.
In this paper, we consider undirected, finite and simple graphs with vertex set $V(\Omega)$ and edge set $E(\Omega)$. In 1966, graph labeling was invented by Rosa in his paper [15]. The most recent Gallian survey [5] indicates that significant work has been devoted to the study of graph labeling. Finding the lucky number of a given graph, however, is a difficult task because it must be proven that there is no lucky labeling with fewer numbers. Czerwinski et al. [4] gave important results about lucky labeling of different graphs in 2009. Two years after Czerwinski's research, Akbari et al. established different useful results regard-ing lucky choice number in their paper [3]. After the span of a year, Ahadi et al. [2] found lucky number of planar graphs. In 2014, Murugan et al. showed that path graph, cycle graph and crown graph admit lucky labeling in their paper [13]. Because of motivation from other authors who were developing useful results regarding lucky number, Kujur et al. computed the lucky number for different families of bloom graph in their work [12] in 2017. After the lapse of one year, Irudhaya et al. computed lucky number of some star related graphs [8]. In 2021, Sateesh Kumar et al. [16] proved that various families of quadrilateral snake graph admit lucky labeling. In the same year, Philomena et al. established the fact that lucky labeling is applicable on shell graphs, bow graphs, wheel graphs in [14]. In 2022, Kumar et al. determined the lucky number of jewel, comb, fan graphs [9], computed the lucky number of triangular graphs [10], calculated the lucky number for jellyfish, cocktail party, crown graphs in their paper [11]. In 2023, one year after Kumar's research, Ashwini et al. [1] determined the lucky number for complete graphs and complete bipartite graphs, also examined the variation in lucky numbers through the addition of graph Ω with K_n and the removal of an edge from K_n. In 2020, Zhang et al. [17, 18] introduced new families of comb graph, namely comb graphs Ca_w, Cd_w, Ce_w, Cf_w and Cg_w. They also determined the edge irregularity strength of these new families of comb graph. In 2022, Imran et al. [7] introduced two new families of comb graph, comb graph Ct_w and Ch_w, proved that comb graphs Ca_w, Cd_w, Ct_w, Cf_w and Ch_w are cordial graphs. In 2022, Imran et al. [6] proved that some families of snake graph are cordial graphs.
Due to the wide range of labeling applications and motivated by the research efforts of previous scholars, we applied lucky labeling into the context of this paper. we found the exact value of lucky number of graphs such as comb graphs Cf_w, Cg_w, Ch_w, triangular snake graph with and without pendant edges, alternate triangular snake graph with pendant edges.

2 Preliminaries

Now, we provide several fundamental definitions that are utilized within this paper.

Definition 2.1. Graph labeling is the procedure of assignment of integers to the elements of a graph Ω. Here, by elements of a graph Ω, we refer to its edges or vertices.

Definition 2.2. If a graph admits lucky labeling, then it is known as a lucky labeled graph.

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Definition 2.3. An edge is categorized as pendant edge if one of its vertices has degree one.

Definition 2.4. First we define general definition of comb graph, let us consider a path graph P_w, having $w \geq 1$ vertices and $(w$-$1)$ edges. The comb graph Cb_w is defined by $P_w \Theta K_1$. It has $2w$ vertices and $2w$-1 edges. There are many types of comb graph which are discussed in this paper, such as comb graphs Cf_w, Cg_w and Ch_w.

3 Main Results

Comb graph Cf_w

[7, 17] Comb graph Cf_w, $w \geq 2$, can be constructed by vertex set $V(Cf_w) = \{ j_v^r; 1 \leq v \leq 7, 1 \leq r \leq w \}$ and edge set $E(Cf_w) = \{ j_4^r j_4^{r+1}; 1 \leq r \leq w-1 \} \bigcup \{ j_v^r j_{v+1}^r; 1 \leq r \leq w, 1 \leq v \leq 6 \}$.

Theorem 3.1. Comb graph Cf_w is a lucky labeled graph with $\eta(Cf_w)=2$.

Proof. To establish that the comb graph Cf_w possesses a lucky labeling and to determine its corresponding lucky number, define a vertex labeling $\lambda: V(Cf_w) \rightarrow \{1, 2, 3, \ldots, z\}$ as follows:

If w is odd

$$\lambda(j_v^r) = \begin{cases} 1, & \text{if } v = 1,7; 1 \leq r \leq w, odd \\ 2, & \text{if } 2 \leq v \leq 6; 1 \leq r \leq w, odd \end{cases}$$

$$\lambda(j_v^r) = \begin{cases} 1, & \text{if } v = 1,5;\ 2 \leq r \leq w-1, even. \\ 2, & \text{if } v = 2,3,4,6,7;\ 2 \leq r \leq w-1, even \end{cases}$$

Now we evaluate the sum of neighbour vertices denoted by $\mu(j^r)$ as follows:

$$\mu(j_v^r) = \begin{cases} 2 & \text{if } v = 1,7; 1 \leq r \leq w \\ 3 & \text{if } v = 2,6; 1 \leq r \leq w \\ 4 & \text{if } v = 3,5; 1 \leq r \leq w. \end{cases}$$

$$\mu(j_v^r) = \begin{cases} 6 & \text{if } v = 4;\ r = 1, w \\ 7 & \text{if } v = 4;\ 2 \leq r \leq w-1, even \\ 8 & \text{if } v = 4;\ 3 \leq r \leq w-2, odd. \end{cases}$$

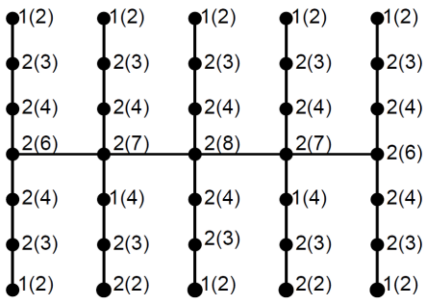

Figure 1:Lucky labeling on comb graph Cf_5.

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If w is even

$$\lambda(j_v^r)=\begin{cases}1 & \text{if } v=1,7; 1\le r\le w-1, odd\\ 2 & \text{if } 2\le v\le 6; 1\le r\le w-1, odd.\end{cases}$$

$$\lambda(j_v^r)=\begin{cases}2 & \text{if } v=2,3,4,6,7; 2\le r\le w, even\\ 1 & \text{if } v=1,5; 2\le r\le w, even.\end{cases}$$

Now we evaluate the sum of neighbour vertices denoted by $\mu(j^r)$ as follows:

$$\mu(j_v^r)=\begin{cases}2 & \text{if } v=1,7; 1\le r\le w\\ 3 & \text{if } v=2,6; 1\le r\le w\\ 4 & \text{if } v=3,5; 1\le r\le w.\end{cases}$$

$$\mu(j_v^r)=\begin{cases}6 & \text{if } v=4; r=1\\ 5 & \text{if } v=4; r=w\\ 7 & \text{if } v=4; 2\le r\le w-2, even\\ 8 & \text{if } v=4; 3\le r\le w-1, odd.\end{cases}$$

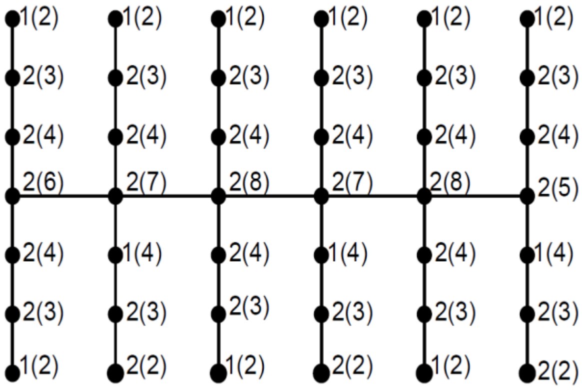

Figure 2:Lucky labeling on comb graph Cf_6.

We see that $S(x)\neq S(y)$ for every pair of adjacent vertices x and y. Hence comb graph Cf_w is a lucky labeled graph. We observe that all vertex labels are at most 2, which implies that there cannot be a lucky number less than 2, 2 is least label for which Cf_w admits lucky labeling. Consequently, we can deduce that $\eta(Cf_w)$=2. This brings us to the end of the proof.■

Comb graph Cg_w

[17] Comb graph Cg_w, ($w\geq 3$, odd), can be constructed by vertex set $V(Cg_w)=\{j_v^r;1\le v\le r+1,1\le r\le \lceil \frac{w}{2}\rceil\} \bigcup \{j_v^r;\lceil \frac{w}{2}\rceil < r\le w, 1\le v\le w-r+2\}$ and edge set

$$E(Cg_w)=\{j_1^r j_1^{r+1};1\le r\le w-1\}\bigcup\{j_v^r j_{v+1}^r;1\le r\le \lceil \frac{w}{2}\rceil, 1\le v\le r\} \bigcup\{j_v^r j_{v+1}^r;\lceil \frac{w}{2}\rceil < r\le w, 1\le v\le w-r+1\}.$$

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Theorem 3.2. Comb graph Cg_w is a lucky labeled graph with $\eta(Cg_w)$=2.

Proof. To establish that the comb graph Cg_w possesses a lucky labeling and to determine its corresponding lucky number, define a vertex labeling $\lambda: V(Cg_w) \to \{1, 2, 3, ..., z\}$ as follows:

If r is odd

$$\lambda(j_v^r) = \begin{cases} 1 & \text{if } 1 \le v \le r+1, odd; 1 \le r \le \left\lceil \frac{w}{2} \right\rceil, odd \\ 2 & \text{if } 2 \le v \le r+1, even; 1 \le r \le \left\lceil \frac{w}{2} \right\rceil, odd. \end{cases}$$

$$\lambda(j_v^r) = \begin{cases} 1 & \text{if } 1 \le v \le w-r+1, odd; \left\lceil \frac{w}{2} \right\rceil < r \le w, odd \\ 2 & \text{if } 2 \le v \le w-r+2, even; \left\lceil \frac{w}{2} \right\rceil < r \le w, odd. \end{cases}$$

If r is even

$$\lambda(j_v^r) = \begin{cases} 1 & \text{if } 2 \le v \le r+1, even; 2 \le r \le \left\lceil \frac{w}{2} \right\rceil, even \\ 2 & \text{if } 1 \le v \le r+1, odd; 2 \le r \le \left\lceil \frac{w}{2} \right\rceil, even. \end{cases}$$

$$\lambda(j_v^r) = \begin{cases} 1 & \text{if } 2 \le v \le w-r+2, even; \left\lceil \frac{w}{2} \right\rceil < r \le w, even. \\ 2 & \text{if } 1 \le v \le w-r+1, odd; \left\lceil \frac{w}{2} \right\rceil < r \le w, even \end{cases}$$

Now we evaluate sum of neighbour vertices denoted by $\mu(j^r)$ as follows:

$$\mu(j_v^r) = \begin{cases} 1 & \text{if } v = r+1; 1 \le r \le \left\lceil \frac{w}{2} \right\rceil \\ 1 & \text{if } v = w-r+2; \left\lceil \frac{w}{2} \right\rceil < r \le w. \end{cases}$$

If r is odd

$$\mu(j_v^r) = \begin{cases} 4 & \text{if } v = 1; r = 1, w \\ 6 & \text{if } v = 1; 3 \le r \le w-2, odd \\ 2 & \text{if } 2 \le v \le w-r, even; 3 \le r \le w, odd \\ 4 & \text{if } 3 \le v \le w-r+1, odd; 3 \le r \le w, odd. \end{cases}$$

If r is even

$$\mu(j_v^r) = \begin{cases} 3 & \text{if } v = 1; 2 \le r \le w-1, even \\ 4 & \text{if } 2 \le v \le w-r+1, even; 2 \le r \le w-1, even \\ 2 & \text{if } 3 \le v \le w-r, odd; 2 \le r \le w-1, even. \end{cases}$$

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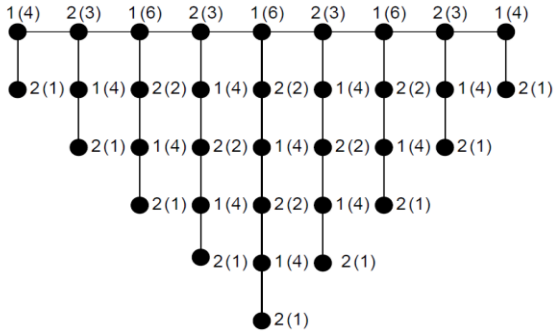

Figure 3:Lucky labeling on comb graph Cg_9.

We see that $S(x) \neq S(y)$ for every pair of adjacent vertices x and y. Hence comb graph Cg_w is a lucky labeled graph. We observe that all vertex labels are at most 2, which implies that

there cannot be a lucky number less than 2, 2 is least label for which Cg_w admits lucky labeling. Consequently, we can deduce that $\eta(Cg_w)=2$. We have now reached the conclusion of the proof.■

Comb graph Ch_w

[7] Comb graph Ch_w, ($w \geq 2$, even), can be constructed by vertex set $V(Ch_w) = \{ j_v^r; 1 \leq v \leq 3, 1 \leq r \leq w, odd\} \bigcup \{ j_v^r; 2 \leq r \leq w, even, 1 \leq v \leq 4\}$ and edge set $E(Ch_w) = \{ j_v^r j_{v+1}^r; 1 \leq r \leq w, odd, 1 \leq v \leq 2\} \bigcup \{ j_v^r j_{v+1}^r; 2 \leq r \leq w, even, 1 \leq v \leq 3\} \bigcup \{ j_1^r j_1^{r+1}; 1 \leq r \leq w-1\}$.

Theorem 3.3. Comb graph Ch_w is a lucky labeled graph with $\eta(Ch_w)=2$.

Proof. To establish that the comb graph Ch_w possesses a lucky labeling and to determine its corresponding lucky number, define a vertex labeling $\lambda: V(Ch_w) \rightarrow \{1, 2, 3, \ldots, z\}$ as follows:

$$\lambda(j_v^r) = \begin{cases} 1 & \text{if } v = 2; 1 \leq r \leq w-1, odd \\ 2 & \text{if } v = 1,3; 1 \leq r \leq w-1, odd \end{cases}$$

$$\lambda(j_v^r) = \begin{cases} 1 & \text{if } v = 1,3; 2 \leq r \leq w, even \\ 2 & \text{if } v = 2,4; 2 \leq r \leq w, even. \end{cases}$$

Now we evaluate the sum of neighbour vertices denoted by $\mu(j^r)$, as follows:

$$\mu(j_v^r) = \begin{cases} 4 & \text{if } v = 2; 1 \leq r \leq w-1, odd \\ 1 & \text{if } v = 3; 1 \leq r \leq w-1, odd. \end{cases}$$

$$\mu(j_v^r) = \begin{cases} 2 & \text{if } v = 2; 2 \leq r \leq w, even \\ 4 & \text{if } v = 3; 2 \leq r \leq w, even \\ 1 & \text{if } v = 4; 2 \leq r \leq w, even. \end{cases}$$

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$$\mu(j_v^r)=\begin{cases}2 & \text{if } v=1; r=1\\ 4 & \text{if } v=1; r=w\\ 6 & \text{if } v=1; 2\le r\le w-2, even\\ 3 & \text{if } v=1; 3\le r\le w-1, odd.\end{cases}$$

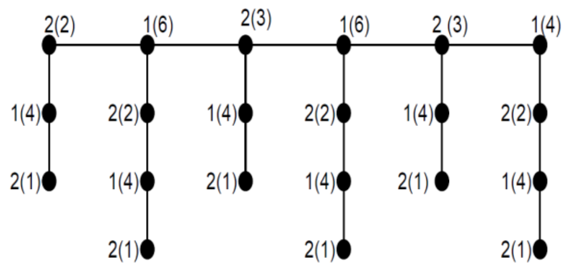

Figure 4:Lucky labeling on comb graph Ch_6.

We see that $S(x)\neq S(y)$ for every pair of adjacent vertices x and y. Hence comb graph Ch_w is a lucky labeled graph. We observe that all vertex labels are at most 2, which implies that there cannot be a lucky number less than 2, 2 is least label for which Ch_w admits lucky labeling. Consequently, we can deduce that $\eta(Ch_w)=2$. It completes the proof.■

Triangular snake graph T_w

[6] Triangular snake graph T_w can be obtained by replacing each edge of path graph P_w,($w>1$), by a triangle C_3. It is constructed by vertex set $V(T_w)=\{j_v;1\le v\le w-1\}\cup\{k_v:1\le v\le w\}$ and edge set $E(T_w)=\{k_vk_{v+1}:1\le v\le w-1\}\cup\{j_vk_v:1\le v\le w-1\}$.

Theorem 3.4. Triangular snake graph T_w, $w>1$, is a lucky labeled graph with

$$\eta(T_w)=\begin{cases}3 & \text{if } w=2\\ 2 & otherwise.\end{cases}$$

Proof. **If w=2**

To establish that the triangular snake graph T_2 possesses a lucky labeling and to deter-mine its corresponding lucky number, define a vertex labeling $\lambda:V(T_2)\rightarrow\{1, 2, 3,..., z\}$ as follows:

$\lambda(j_1)=1$.

$\lambda(k_v)=v+1$, $1\le v\le 2$.

Now we evaluate the sum of neighbour vertices denoted by μ as follows:

$\mu(j_1)=5$.

$\mu(k_v)=5-v$, $1\le v\le 2$.

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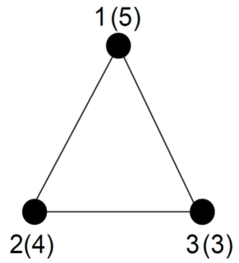

Figure 5:Lucky labeling on triangular snake graph T_2.

We see that $S(x)\neq S(y)$ for every pair of adjacent vertices x and y. Hence triangular snake graph T_2 is a lucky labeled graph. We observe that all vertex labels are at most 3, which implies that there cannot be a lucky number less than 3, 3 is least label for which T_2 admits lucky labeling. Consequently, we can deduce that $\eta(T_2)=3$.

If $w>2$, odd

$\lambda(j_v)=1$, if $1\leq v\leq w-1$.

$$\lambda(k_v)=\begin{cases}2 & \text{if } 1\leq v\leq w, odd\\ 1 & \text{if } 2\leq v\leq w-1, even.\end{cases}$$

Now we evaluate the sum of neighbour vertices denoted by μ as follows:

$\mu(j_v)=3$, if $1\leq v\leq w-1$.

$$\mu(k_v)=\begin{cases}2 & \text{if } v=1,w\\ 6 & \text{if } 2\leq v\leq w-1, even\\ 4 & \text{if } 3\leq v\leq w-2, odd.\end{cases}$$

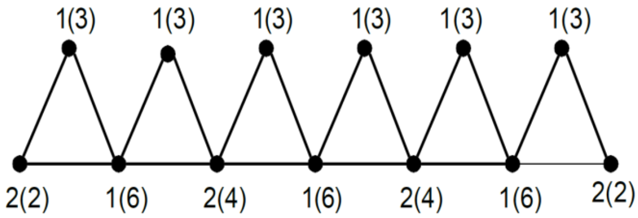

Figure 6:Lucky labeling on triangular snake graph T_7.

If $w>2$, even

$$\lambda(j_v)=\begin{cases}1 & \text{if } 1\leq v\leq w-2\\ 2 & \text{if } v=w-1.\end{cases}$$

$$\lambda(k_v)=\begin{cases}1 & \text{if } 2\leq v\leq w, even\\ 2 & \text{if } 1\leq v\leq w-1, odd.\end{cases}$$

Now we evaluate the sum of neighbour vertices denoted by μ as follows:

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$$\mu(k_v)=\begin{cases}2 & \text{if } v=1\\ 4 & \text{if } v=w\\ 5 & \text{if } v=w-1\\ 6 & \text{if } 2\le v\le w-2, even\\ 4 & \text{if } 3\le v\le w-3, odd.\end{cases}$$

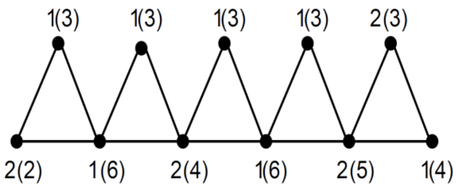

Figure 7:Lucky labeling on triangular snake graph T_6.

We see that $S(x)\neq S(y)$ for every pair of adjacent vertices x and y. Hence triangular snake graph T_w is a lucky labeled graph. We observe that all vertex labels are at most 2, which implies that there cannot be a lucky number less than 2, 2 is least label for which T_w admits lucky labeling. Consequently, we can deduce that $\eta(T_w)$=2. This concludes our proof.

Finally, we can say that:

$$\eta(T_w)=\begin{cases}3 & \text{if } w=2\\ 2 & otherwise.\end{cases} \blacksquare$$

Triangular snake graph with pendant edges T_w

[6] Triangular snake graph Tw, (w>1), with pendant edges can be constructed by vertex set $V(T_w)=\{j_v,k_v;1\le v\le w-1\}\bigcup\{l_v,m_v;1\le v\le w\}$ and edge set $E(T_w)=\{j_vk_v;1\le v\le w-1\}\bigcup\{k_vl_v;1\le v\le w-1\}\bigcup\{k_vl_{v+1};1\le v\le w-1\}$ $\bigcup\{l_vl_{v+1};1\le v\le w-1\}\bigcup\{l_vm_v;1\le v\le w\}$.

Theorem 3.5. Triangular snake graph T_w with pendant edges is a lucky labeled graph with $\eta(T_w)$=2.

Proof. To establish that triangular snake graph T_w with pendant edges possesses a lucky labeling and to determine its corresponding lucky number, define a vertex labeling $\lambda:V(T_w)\rightarrow\{1, 2, 3,..., z\}$ as follows:

If w is odd

$\lambda(j_v)$=1, if $1\le v\le w$-1.

$\lambda(k_v)$=2, if $1\le v\le w$-1

.$\lambda(l_v)$=1, if $1\le v\le w$.

$$\lambda(m_v)=\begin{cases}2 & \text{if } 1\le v\le w, odd\\ 1 & \text{if } 2\le v\le w-1, even.\end{cases}$$

Now we evaluate the sum of neighbour vertices denoted by μ as follows:

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$\mu(j_v)$=2, if $1\leq v\leq w-1$.
$\mu(k_v)$=3, if $1\leq v\leq w-1$.

$$\mu(l_v)=\begin{cases}7 & \text{if } 2\leq v\leq w-1, even\\ 8 & \text{if } 3\leq v\leq w-2, odd\\ 5 & \text{if } v=1,w.\end{cases}$$

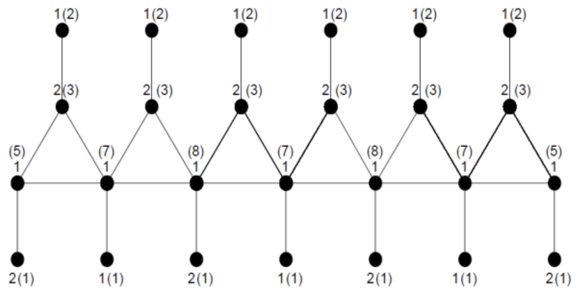

Figure 8:Lucky labeling on triangular snake graph T_7 with pendant edges.

If w is even

$\lambda(j_v)$=1, if $1\leq v\leq w-1$.
$\lambda(k_v)$=2, if $1\leq v\leq w-1$.
$\lambda(l_v)$=1, if $1\leq v\leq w$.

$$\lambda(m_v)=\begin{cases}2 & \text{if } 1\leq v\leq w-1, odd\\ 1 & \text{if } 2\leq v\leq w, even.\end{cases}$$

Now we evaluate the sum of neighbour vertices denoted by μ as follows:
$\mu(j_v)$=2, if $1\leq v\leq w-1$.
$\mu(k_v)$ = 3; if $1\leq v\leq w-1$.

$$\mu(l_v)=\begin{cases}7 & \text{if } 2\leq v\leq w-2, even\\ 8 & \text{if } 3\leq v\leq w-1, odd\\ 5 & \text{if } v=1\\ 4 & \text{if } v=w.\end{cases}$$

$\mu(m_v)$ = 1; if $1\leq v\leq w-1$.

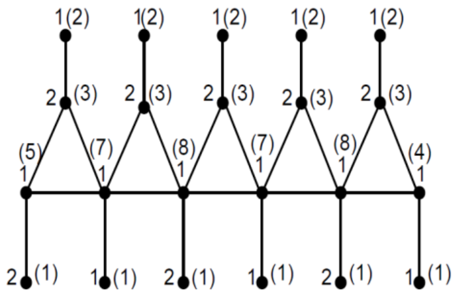

Figure 9:Lucky labeling on triangular snake graph T_6 with pendant edges.

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We see that $S(x) \neq S(y)$ for every pair of adjacent vertices x and y. Hence triangular snake graph T_w with pendant edges is a lucky labeled graph. We observe that all vertex labels are at most 2, which implies that there cannot be a lucky number less than 2, 2 is least label for which T_w admits lucky labeling. Consequently, we can deduce that $\eta(T_w)=2$. It completes the proof.■

Alternate triangular snake graph with pendant edges ATw

[6] Alternate triangular snake graph ATw, w>4, with pendant edges can be constructed by vertex set $V(AT_w)=\{j_v,k_v;1\leq v\leq w-2\}\bigcup\{l_v,m_v;1\leq v\leq w\}$ and edge set $E(AT_w)=\{j_vk_v;1\leq v\leq w-2\}\bigcup\{k_vl_v;1\leq v\leq w-2\}$ $\bigcup\{k_vl_{v+2};1\leq v\leq w-2\}\bigcup\{l_vl_{v+1};1\leq v\leq w-1\}\bigcup\{l_vm_v;1\leq v\leq w\}$.

Theorem 3.6. Alternate triangular snake graph AT_w with pendant edges is a lucky labeled graph with $\eta(AT_w)=2$.

Proof. To establish that the alternate triangular snake graph AT_w with pendant edges possesses a lucky labeling and to determine its corresponding lucky number, define a vertexlabeling $\lambda: V(AT_w)\rightarrow\{1, 2, 3,..., z\}$ as follows:

If $w>4$, **(odd)**

$\lambda(j_v)=1$, if $1\leq v\leq w-2$.

$\lambda(k_v)=2$, if $1\leq v\leq w-2$.

$\lambda(l_v)=1$, if $1\leq v\leq w$.

$$\lambda(m_v)=\begin{cases}2 & \text{if } 1\leq v\leq 3\\ 2 & \text{if } w-2\leq v\leq w\\ 1 & \text{if } 4\leq v\leq w-3, even\\ 2 & \text{if } 5\leq v\leq w-4, odd.\end{cases}$$

Now we evaluate the sum of neighbour vertices denoted by μ as follows:

$\mu(j_v)=2$, if $1\leq v\leq w-2$.

$\mu(k_v)=3$, if $1\leq v\leq w-2$.

$$\mu(l_v)=\begin{cases}5 & \text{if } v=1,w\\ 6 & \text{if } v=2,w-1\\ 8 & \text{if } 3\leq v\leq w-2, odd\\ 7 & \text{if } 4\leq v\leq w-3, even.\end{cases}$$

$\mu(m_v) = 1$; *if* $1\leq v\leq w-1$.

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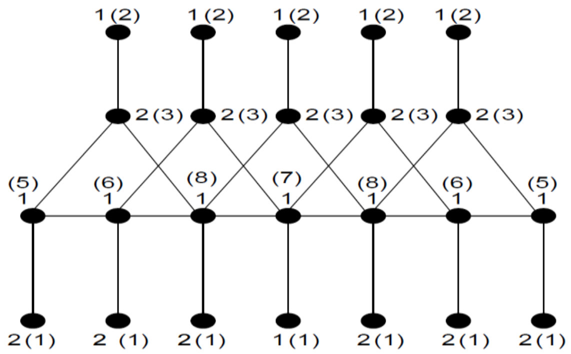

Figure 10:Lucky labeling on alternate triangular snake graph AT_7 with pendant edges.

If $w>4$, **(even)**

$\lambda(j_v)=1$, if $1\le v\le w-2$.

$\lambda(k_v)=2$, if $1\le v\le w-2$.

$\lambda(l_v)=1$, if $1\le v\le w$.

$$\lambda(m_v)=\begin{cases}2 & \text{if } 1\le v\le 3\\ 2 & \text{if } w-1\le v\le w\\ 1 & \text{if } 4\le v\le w-2, even\\ 2 & \text{if } 5\le v\le w-3, odd.\end{cases}$$

Now we evaluate the sum of neighbour vertices denoted by μ as follows:

$\mu(j_v)=2$, if $1\le v\le w-2$.

$\mu(k_v)=3$, if $1\le v\le w-2$.

$$\mu(l_v)=\begin{cases}5 & \text{if } v=1,w\\ 6 & \text{if } v=2,w-1\\ 8 & \text{if } 3\le v\le w-3, odd\\ 7 & \text{if } 4\le v\le w-2, even.\end{cases}$$

$\mu(m_v) = 1;$ *if* $1\le v\le w-1$.

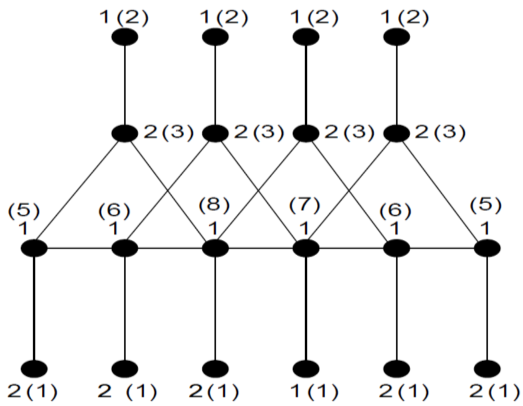

Figure 11:Lucky labeling on alternate triangular snake graph AT_6 with pendant edges.

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We see that $S(x) \neq S(y)$ for every pair of adjacent vertices x and y. Hence alternate triangular snake graph AT_w with pendant edges is a lucky labeled graph. We observe that all vertex labels are at most 2, which implies that there cannot be a lucky number less than 2, 2 is least label for which AT_w admits lucky labeling. Consequently, we can deduce that $\eta(AT_w) = 2$. It completes the proof.■

4 Conclusion

In this study, we demonstrated that comb graphs Cf_w, Cg_w, Ch_w, along with different families of snake graphs, admit lucky labeling. Additionally, we have successfully determined the exact value of lucky number for the aforementioned graphs. Our findings not only contribute to the existing body of knowledge in the field but also shed light on the underlying properties of these graphs. Through rigorous analysis and computation, we have established a clear understanding of their unique characteristics. This research not only adds to the current literature but also opens avenues for further exploration and deeper insights into the realm of graph theory.

Data Availability

In this article, no data were utilized.

Conflicts of Interest

Authors have no conflict of interest. All authors have read and agreed to the published version of the manuscript.

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