Power System Technology
ISSN:1000-3673

Accepted: 07-03-2024 *Revised: 12-02-2024* *Received: 16-01-2024*

Buckling Analysis of a Neo-Hookean Elastic Cylindrical Shell by WKB Method

Murteza Sanjaranipour*

*Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

E-mail: sanjarani@math.usb.ac.ir or msanjarani@yahoo.co.uk

Abstract

Our focus in this article is to consider the axisymmetric deformation of an incompressible, isotropic thick-walled circular cylindrical shell. This neo-Hookean elastic shell is subjected to combined axial loading and internal pressure. For finding the rate of change of the inner radius ($\lambda_a = a/A$) and the outer radius ($\lambda_b = b/B$), concerning the variations of the wall thickness (A/B) for different mode numbers and axial stretches, the incremental equilibrium equations derived by Haughton and Ogden are solved with the asymptotic WKB method, (where A and B are the un-deformed and a and b are the deformed inner and outer radii respectively). The graph of the eigenvalues which shows the radius changes concerning the changes of the wall thickness has been plotted. Our derived asymptotic results are shown to agree with the counterpart data obtained by using the numerical compound matrix method.

Keywords: Compound Matrix method, Eigen-values, neo-Hookean material, WKB method.

1. Introduction

To check that any elastic structure becomes unstable or not under excessive loads a linear stability analysis should be applied. Using the normal mode in such analysis results always in an eigenvalue problem, the solution of which gives a bifurcation condition. The eigenvalues problems except for a few simple problems usually have to be solved by a full numerical integration. As the mode number increases, integration becomes difficult for such problems.

The asymptotic WKB method has been widely applied to the buckling and stability analysis of elastic problems with different geometries. The asymptotic results may also be applied to validate the numerical code or can be used as the initial guess in fully numerical integrations. The stability of cylindrical tubes has been analyzed numerically and asymptotically by different researchers. Wilkes [21], initially considered the stability of a circular tube under end trust and showed that the linear system around a finite axial strain can be solved exactly in terms of the Bessel functions. Green and Spencer [6] considered the stability of the same tube under finite extension and torsion. Sierakowski et al. [20] have investigated the axisymmetric modes of deformation of a neo-Hookean elastic tube subjected to internal pressure and axial compression. The detailed results of Haughton and Ogden [9] are about the bifurcation of a circular cylindrical membrane tube subjected to the combined axial loading and internal pressure. In part II of this series, Haughton and Ogden [10] considered the prismatic, axisymmetric, and asymmetric bifurcations for the Three-Term strain energy tubes of finite wall thickness under axial tension and compression combined with internal and

Accepted: 07-03-2024 *Revised: 12-02-2024* *Received: 16-01-2024*

external pressures. The fourth–order Runge–Kutta method has been applied for the solution of the eigen-value problem. For the first time, Fu [3] applied the WKB method on solving the finite elastic eigen-value problems. He considered the asymptotic bifurcation analysis of a neo-Hookean spherical elastic shell of arbitrary thickness under external pressure. Both Fu and Sanjaranipour [5] and Fu and Lin [4] respectively applied this method to the stability analysis of Varga and neo-Hookean everted cylindrical tubes. Haughton and Chen [8] used the same method for the bifurcation analysis of an everted cylindrical shell for both compressible and incompressible Varga materials. The buckling condition for Varga cylindrical shell of arbitrary thickness also has been analyzed by Sanjaranipour [16].

A more general bifurcation description of thick-walled tubes subjected to the combined axial loading and external pressure has been analyzed by Zhu et al. [22]. They analyzed the asymmetric bifurcation and solved the eigenvalue problem by Adams-Moulton method for a specific elastic material. Coman and Destrade [2] employed the WKB method for solving the problem of pure bending of a neo-Hookean elastic rubber block. Sanjaranipour et al. [19 solved the elastic eigen-value problem of a Varga cube under the bifurcation criterion and obtained the bending angles. Sanjaranipour and Irandegani [18] analyzed the axisymmetric bifurcation of a Three-term elastic cylindrical shell under axial loading and external pressure by Adams-Moulton method and Compound Matrix method. Abdulalian and Sanjaranipour [1] considered the deformation of a circular cylindrical tube under rotation about its axis for two cases i.e. with and without the axial force and obtained the angular speed as a function of an azimuthal deformation parameter for the neo-Hookean strain energy function. Liu [12] used the WKB method for obtaining the eigenvalue results from the deformation of a circular cylindrical shell under finite pressure and fixed outer radius and for high mode number. They found that the WKB results are coincident with the numerical data for the high mode numbers. Omugbe et al. [14] used the WKB method to solve the Schrödinger equation with the Killingbeck potential plus an inversely quadratic potential (KPIQP) function. They obtained the energy eigenvalues and the mass spectra of the heavy and heavy-light mesons systems. Sahu and Nirwal [15] used the WKB method for the propagation of Love-type surface waves in a smart composite structure involving functionally graded piezoelectric material (FGPM). They derived Frequency equations for electrically and magnetically open and short circuit cases. Gristchak et al. [7] analyzed the stability of a 3-layered conical elastic shell which is under the combined loading.

They solved the six-order differential equation by WKB method compared the obtained results with the numerical data and observed a good agreement.

Our main focus in this work is to solve the system of equations (by using the system form of the WKB method [16]). To validate our obtained asymptotic solution we also solved the eigenvalue problem by using the Compound Matrix method. Comparing the results of the mentioned methods shows excellent agreement. Finally, the graph which shows the changes in the radius concerning the changes in the wall thickness has been plotted. For the numerical finding, we used Fortran 90 programming (Metcalf et al., [13]) and to obtain the asymptotic solutions, the Mathematica software has been employed.

Power System Technology
ISSN:1000-3673

Accepted: 07-03-2024 *Revised: 12-02-2024* *Received: 16-01-2024*

2 The finitely-deformed circular cylindrical configuration

It was hypothesized that the thick-walled cylindrical shell is defined by Haughton and Orr [11] as**:**

$$A \leq R \leq B, \; 0 \leq \Theta \leq 2\pi, \; 0 \leq Z \leq L, \tag{1}$$

where, R, Θ and Z are the cylindrical polar coordinates, A and B are the inner and outer radii of the shell respectively and L is the length all in an un-deformed configuration. The initial deformed configuration of the cylindrical shell under the action of the axial loading and internal pressure is assumed also to be a circular cylindrical shell with the following geometry:

$$a \leq r \leq b, \; 0 \leq \theta \leq 2\pi, \; 0 \leq z \leq l, \tag{2}$$

where, r, θ and z are the cylindrical polar coordinates, a and b are respectively the inner and outer radii and l is the length of the shell in the current configuration. Since the material is incompressible, the deformation is described by the following equations:

$$r^2 = \lambda_z^{-1}(R^2 - A^2) + a^2, \; \theta = \Theta, \; z = \lambda_z Z, \tag{3}$$

where, λ_z $(= l/L)$ is the axial extension (stretch) uniform ratio. We use e_1, e_2 and e_3 to denote the unit basis vectors corresponding to the coordinates (θ, z, r) respectively and let λ_1, λ_2 and λ_3 be the corresponding principal stretches. In view of the relation $(3)_1$ and the incompressibility constraint det($\boldsymbol{F}$)=$\lambda_1\lambda_2\lambda_3$= 1, we have:

$$\lambda_2 = \lambda_z, \; \lambda_1 = r/R, \; \lambda_3 = (\lambda_1\lambda_2)^{-1}, \tag{4}$$

where $\boldsymbol{F}$ is the deformation gradient and $R = \sqrt{A^2 + \lambda_z(r^2 - a^2)}$. Given $(4)_2$ the eigenvalues are defined by λ_a =a/A and $\lambda_b = b/B$. For incompressible material, the equilibrium equations reduce to the following single equation:

$$r\frac{d\sigma 33}{dr} + \sigma_{33} - \sigma_{11} = 0, \tag{5}$$

and we have the associated boundary conditions as

$$\sigma_{33} = \begin{cases} -p & r = a, \\ 0 & r = b, \end{cases} \tag{6}$$

where, p denote the internal hydrostatic pressure and also the principal Cauchy stretches are given by:

$$\sigma_{ii} = \sigma_i - p = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, i = 1, \; 2, \; 3, \tag{7}$$

and W=W $(\lambda_1, \; \lambda_2, \; \lambda_3)$ is the strain-energy function which for the neo-Hookean materials is:

$$W = \frac{1}{2}({\lambda_1}^2 + {\lambda_2}^2 + {\lambda_3}^2 - 3). \tag{8}$$

3 Analysing the Bifurcation criterion

A brief description of the equilibrium equation and the relevant boundary conditions given the derivation of Haughton and Ogden [10] has been given in the following part. In the absence of the body forces the incremental equilibrium equations are:

$$div\, \boldsymbol{\chi} = 0, \tag{9}$$

where, *div* is the divergence operator and $\boldsymbol{\chi}$ is the increment of the nominal stress, both in the current configuration. The incremental boundary conditions for the hydrostatic pressure loading are:

$$\boldsymbol{\chi}^T \boldsymbol{n} = p\, \boldsymbol{V}^T \boldsymbol{n} - \dot{p}\, \boldsymbol{n}, \tag{10}$$

Power System Technology
ISSN:1000-3673
Accepted: 07-03-2024 Revised: 12-02-2024 Received: 16-01-2024

which is evaluated on the appropriate boundary, where, superscript T denotes the transpose, $\boldsymbol{n}$ is the unit normal in the current configuration and $\boldsymbol{V}$ is the incremental deformation gradient. The incremental constitutive law is:

$$\boldsymbol{\chi} = \boldsymbol{B}\,\boldsymbol{V} + p\,\boldsymbol{V}^{\boldsymbol{T}} - \dot{p}\,\boldsymbol{I}, \tag{11}$$

where $\boldsymbol{B}$ is the fourth-order tensor of instantaneous moduli in the current configuration and $\boldsymbol{I}$ is the identity tensor. The non-zero components of $\boldsymbol{B}$ for a general isotropic material which has been given by Haughton and Orr [11] are:

$$\left.\begin{aligned} B_{ijij} &= \lambda_i^2 \frac{\sigma_i-\sigma_j}{\lambda_i^2-\lambda_j^2}, \quad \lambda_i \neq \lambda_j, \\ B_{iijj} &= B_{jjii} = \lambda_i\lambda_j \frac{\partial^2 w}{\partial\lambda_i\partial\lambda_j}, \\ B_{ijji} &= B_{jiij} = B_{ijij} - \sigma_i, \; i \neq j, \end{aligned}\right\} \tag{12}$$

where, λ_i and λ_j $(i,j = 1,2,3)$ are introduced in the previous section. By considering the incremental displacement $\dot{\boldsymbol{u}} = (u(\theta,z,r), v(\theta,z,r), w(\theta,z,r))$, the components of $\boldsymbol{V} = \mathrm{grad}(\dot{\boldsymbol{u}})$ are displayed by:

$$\boldsymbol{V} = \begin{bmatrix} (u+v_\theta)/r & v_z & v_r \\ w_\theta/r & w_z & w_r \\ (u_\theta - v)/r & u_z & u_r \end{bmatrix}, \tag{13}$$

where subscripts here denote the partial derivatives. The incremental form of the incompressibility condition is:

$$tr(\boldsymbol{V}) \equiv u_r + (u+v_\theta)/r + w_z = 0. \tag{14}$$

By using relations (9)-(14) and given the main governing equations of Haughton and Orr [11], we have:

$$\dot{p}_\theta = (rB'_{3131} + B_{3131})(u_\theta + r\,v_r - v)/r + (B_{1111} - B_{1122} - B_{2112})(u_\theta + v_{\theta\theta})/r + B_{2121} r\,v_{zz} + B_{3131} r\,v_{rr} + (B_{1133} - B_{1122} - B_{3131} - B_{2112} + B_{3113})\,u_{r\theta}, \tag{15}$$

$$\dot{p}_z = (r\,B'_{3232} + B_{3232})(u_z + w_r)/r + B_{3232}w_{rr} + B_{1212}(w_{\theta\theta} - r\,u_z)/r^2 + (B_{2222} - B_{1221} - B_{1122})\,w_{zz} + (B_{2233} + B_{3223} - B_{1221} - B_{1122})\,u_{rz}, \tag{16}$$

$$\dot{p}_r = (r\,B'_{1133} - r\,B'_{2233} - B_{1111} + B_{1122})(u+v_\theta)/r^2 + B_{1313}(u_{\theta\theta} - v_\theta)/r^2 + B_{3223}w_{rz} + (B_{1331} + B_{1133} - B_{2233})v_{r\theta}/r + (B_{3333} - B_{2233})u_{rr} + B_{2323}u_{zz} + (r\,B'_{3333} + r\,p' - r\,B'_{2233} + B_{3333} - 2B_{2233} + B_{1122})u_r/r, \tag{17}$$

and by using relations (10), (11), and (14) we obtain the following boundary conditions:

$$\left.\begin{aligned} u_\theta + r\,v_r - v &= 0, \\ u_z + w_r &= 0, \\ (B_{3333} + p)u_r + B_{1133}\frac{1}{r}(u+v_\theta) + B_{2233}w_z - \dot{p} &= 0. \end{aligned}\right\} (r = a, b)$$

In order to solve the equations (15)-(17), we apply the following form of solutions:

$$\left.\begin{aligned} u &= f(r)\,cos(m\,\theta)\,sin(\alpha\,z), \\ v &= g(r)\,sin(m\,\theta)\,sin(\alpha\,z), \\ w &= h(r)cos\,(m\,\theta)\,cos\,(\alpha\,z), \\ \dot{p} &= k(r)\,cos(m\,\theta)\,sin(\alpha\,z), \end{aligned}\right\} \tag{18}$$

Accepted: 07-03-2024 Revised: 12-02-2024 Received: 16-01-2024

where, m $(= 0, 1, 2, \ldots)$, is the azimuthal mode number. The parameter α and the associated end conditions can be interpreted in different ways. But here due to equations (18), we may conclude that the incremental end displacement w should be zero on the ends. This leads to:

$$\alpha = \frac{n\,\pi}{l}, \qquad (n = 1, 2, 3, \ldots),$$

where n is the axial mode number. To be more precise about n and the relevant parameters, we refer the reader to the paper of Haughton and Orr [11], Substitution (18) into the incompressibility condition (14), yield:

$$r\,f'(r) + f(r) + m\,g(r) - \alpha\,r\,h(r) = 0. \tag{19}$$

By applying (18) in (15)-(17) and in view of (19), $h(r)$ can be eliminated and finally three coupled equations for $f(r)$, $g(r)$ and $k(r)$ are obtained as:

$$(r\,B'_{3131} + B_{3131} + B_{1111} - B_{1122} - B_{2112})m\,f(r) + (B_{1133} - B_{1122} + B_{3113} - B_{2112})m\,r\,f'(r) + [r\,B'_{3131} + B_{3131} + \alpha^2 r^2 B_{2121} + m^2(B_{1111} - B_{1122} - B_{2112})]\,g(r) + (r\,B'_{3131} + B_{3131})r\,g'(r) - B_{3131} r^2 g''(r) - m\,r\,k(r) = 0, \tag{20}$$

$$[r\,B'_{3232} - B_{3232} + m^2 B_{1212} - \alpha^2 r^2 (r\,B'_{3232} + B_{3232} - B_{1212} + B_{1122} + B_{1221} - B_{2222})]f(r) - [rB'_{3232} - B_{3232} - m^2 B_{1212} - \alpha^2 r^2 (B_{2222} - B_{2233} - B_{3223})]r\,f'(r) - r\,B'_{3232} + 2B_{3232})r^2 f''(r) - B_{3232} r^3 f'''(r) + [r\,B'_{3232} - B_{3232} + m^2 B_{1212} + \alpha^2 r^2 (B_{2222} - B_{1122} - B_{1221})]m\,g(r) - (r\,B'_{3232} - B_{3232})m\,r\,g'(r) - B_{3232} m\,r^2 g''(r) + \alpha^2 r^3 k(r) = 0, \tag{21}$$

$$(r\,B'_{1133} - r\,B'_{2233} - B_{1111} + B_{1122} + B_{3223} - m^2 B_{1313} - \alpha^2 r^2 B_{2323})f(r) + (r\,B'_{3333} + r\,p' - r\,B'_{2233} + B_{3333} - 2B_{2233} + B_{1122} - B_{3223})r\,f'(r) + (B_{3333} - B_{2233} - B_{3223})r^2 f''(r) + (r\,B'_{1133} - r\,B'_{2233} - B_{1111} + B_{1122} + B_{3223} - B_{1313})m\,g(r) + (B_{1133} - B_{2233} + B_{1331} - B_{3223})m\,r\,g'(r) - r^2 k'(r) = 0. \tag{22}$$

The corresponding boundary conditions on the curved surface are given by:

$$\left.\begin{aligned} &r\,g'(r) - g(r) - m\,f(r) = 0,\\ &r^2 f''(r) + r\,f'(r) + (\alpha^2 r^2 + m^2 - 1)f(r) = 0,\\ &(B_{1133} - B_{2233})\big(f(r) + m\,g(r)\big) + (B_{3333} - B_{2233} + \sigma_3)r\,f'(r) - r\,k(r) = 0, \end{aligned}\right\} (r = a, b) \tag{23}$$

where $\sigma_3 = \frac{\partial W}{\partial \lambda_3}$ and m is the azimuthal mode number.

4 The solution of the eigenvalue problem by WKB method

The following form of the WKB method [5] has been applied for the solution of the eigenvalue problem (20)-(22) and the relevant boundary conditions for the case of $m > 1$

$$\begin{Bmatrix} f\\ g\\ k \end{Bmatrix} = \begin{Bmatrix} F\\ G\\ m\,K \end{Bmatrix} exp(m\int_a^r s(x)dx), \tag{24}$$

where,

$$\left.\begin{aligned} F &= F_0 + F_1/m + F_2/m^2 + \cdots,\\ G &= G_0 + G_1/m + G_2/m^2 + \cdots,\\ K &= K_0 + K_1/m + K_2/m^2 + \cdots, \end{aligned}\right\} \tag{25}$$

Power System Technology
ISSN:1000-3673
Accepted: 07-03-2024 Revised: 12-02-2024 Received: 16-01-2024

and F, G, K, and s are functions of r and have to be determined. Substituting (24) and (25) into the incremental equilibrium equations (20)-(22) and collecting the coefficients of the like powers of m by using Mathematica, to the leading order, we obtain:

$$\left.\begin{aligned} G_0(B_{1111}-r^2B_{3131}s^2)-r\,K_0=0,\\ F_0(r^3B_{3232}s^3-r\,B_{1212}s)+G_0(r^2B_{3232}s^2-B_{1212})=0,\\ F_0(r^2B_{3333}s^2-B_{1313})-r^2s\,K_0=0.\end{aligned}\right\} \quad (26)$$

A non-trivial solution for $\{F_0,\ G_0,\ K_0\}^T$ requires that the determinant of the coefficient matrix vanish and we have:

$$-(1-r^2s^2)(B_{1212}-B_{3232}r^2s^2)^2=0,$$

which yield:

$$\left.\begin{aligned} s^{(1)}=\frac{1}{r},\ s^{(2)}=-\frac{1}{r},\\ s^{(3)}=s^{(5)}=\frac{\sqrt{B_{1212}}}{r\sqrt{B_{3232}}},\ s^{(4)}=s^{(6)}=-\frac{\sqrt{B_{1212}}}{r\sqrt{B_{3232}}}.\end{aligned}\right\} \quad (27)$$

Our immediate task is to find six independent solutions for $\{F,\ G,\ K\}^T$. We use $\{F^{(i)},\ G^{(i)},\ K^{(i)}\}^T$, $(i=1,\dots,6)$ to denote the obtained solutions and assume that the superscripts here correspond to those in (27).

Corresponding to (25), we have the expansions:

$$\left.\begin{aligned} F^{(i)}=F_0{}^{(i)}+F_1{}^{(i)}/m+F_2{}^{(i)}/m^2+\cdots,\\ G^{(i)}=G_0{}^{(i)}+G_1{}^{(i)}/m+G_2{}^{(i)}/m^2+\cdots,\\ K^{(i)}=K_0{}^{(i)}+K_1{}^{(i)}/m+K_2{}^{(i)}/m^2+\cdots.\end{aligned}\right\} \quad (28)$$

Given these expansions, relations (26), (for $i=1,\dots,6$) becomes:

$$G_0{}^{(i)}(B_{1111}-r^2B_{3131}s^2)-r\,K_0{}^{(i)}=0, \quad (29)$$

$$F_0{}^{(i)}(r^3B_{3232}\,s^3-r\,B_{1212}\,s)+G_0{}^{(i)}(r^2B_{3232}s^2-B_{1212})=0, \quad (30)$$

$$F_0{}^{(i)}(r^2B_{3333}s^2-B_{1313})-r^2s\,K_0{}^{(i)}=0. \quad (31)$$

By collecting the first-order terms of m from (20)-(22) of course for the non-repeated roots $(27)_1$, yield:

$$\left.\begin{aligned} G_1{}^{(i)}=-F_0{}^{(i)}-r\,F_1{}^{(i)}s^{(i)}-r\,F'_0{}^{(i)},\\ K_1{}^{(i)}=F_1{}^{(i)}s^{(i)}(-B_{1212}+B_{3232})\\ -(B_{1212}-3B_{3232})F'_0{}^{(i)}+2\,F_0{}^{(i)}(B_{3232}+r\,B'_{3232})/r.\end{aligned}\right\} \quad (32)$$

Given Fu and Sanjaranipour (Fu and Sanjaranipour, 2002) and by substituting (4), (12), and $(27)_1$ into the equations (32) and with some simplification by using Mathematica, we obtained the equation of F_0 as:

$$F'_0{}^{(i)}-\frac{Q_1+Q_2}{r\,(Q_3+Q_4)}F_0{}^{(i)}=0, \qquad (i=1,2),$$

where,

$$Q_1=A^4+2A^2(-a^2+r^2)\,\lambda_z,$$

$$Q_2=(a^4-2a^2r^2+2r^4)\,\lambda_z{}^2,$$

Power System Technology
ISSN:1000-3673
Accepted: 07-03-2024 Revised: 12-02-2024 Received: 16-01-2024

$Q_3 = A^4 + 2A^2(-2a^2 + 3r^2)\lambda_z,$
$Q_4 = (a^4 - 3a^2r^2 + 2r^4)\,\lambda_z{}^2,$
and the solutions are:

$$F_0{}^{(i)} = \frac{r\sqrt{A^2 + (-a^2 + r^2)\lambda_z}}{\sqrt{A^2 - (a^2 - 2r^2)\lambda_z}}, \qquad (i = 1, 2).$$

Similarly, by using the same procedure we can find the equations of G_0 and K_0 and obtain respectively the solutions:

$$G_0{}^{(i)} = \frac{-r\sqrt{A^2+(-a^2+r^2)\lambda_z}}{\sqrt{A^2-(a^2-2r^2)\lambda_z}}, \qquad (i = 1, 2),$$

$$K_0{}^{(1)} = -K_0{}^{(2)} = \frac{(A^2-a^2\lambda_z)\sqrt{A^2-(a^2-2r^2)\lambda_z}}{r^2\,\lambda_z{}^2\sqrt{A^2+(-a^2+r^2)\lambda_z}}.$$

By continuing the same procedure, we can easily obtain the functions $F_1{}^{(i)}, G_1{}^{(i)}, K_1{}^{(i)}, F_2{}^{(i)}, G_2{}^{(i)}, K_2{}^{(i)}$, (for $i = 1, 2$). It is necessary to mention that the differential equations related to the above functions are lengthy and are not necessary to be written down here.

We now deal with the repeated roots (i.e. for $s^{(3)} = s^{(5)},\ s^{(4)} = s^{(6)}$). By collecting the first-order terms of (20)-(22), we obtain the equation of F_0 as:

$$F'_0{}^{(i)} - \left\{\frac{2r^3\lambda_z{}^2}{A^4 + A^2(-2a^2 + 3r^2)\lambda_z + (a^4 - 3a^2\,r^2 + 2r^4)\lambda_z{}^2}\right\} F_0{}^{(i)} = 0, (i = 3, 4, 5, 6),$$

where the solutions are:

$$F_0{}^{(3)} = F_0{}^{(4)} = \frac{-(A^2 + (-a^2 + r^2)\lambda_z}{\sqrt{A^2 - (a^2 - 2r^2)\lambda_z}}, F_0{}^{(5)} = F_0{}^{(6)} = 0.$$

We have found the equations and the relevant solutions of $G_0{}^{(i)}$, $K_0{}^{(i)}$, $F_1{}^{(i)}$, $G_1{}^{(i)}$, $K_1{}^{(i)}$, $F_2{}^{(i)}, G_2{}^{(i)}, K_2{}^{(i)}, (i = 3, 4, 5, 6)$ as before. But since are very lengthy (like the previous case) it is not necessary to be written down here.

Given the results established (for the repeated and non-repeated roots), we are now in the position to write down the general solution:

$$\{f, g, k\}^T = \sum_{i=1}^{6} c_i \left\{ F^i, G^i, m\, K^i \right\}^T E^i, \tag{33}$$

where

$$E^{(i)} = exp\left(m\int_a^r s^i(x)dx\right),$$

and c_i, $(i = 1, 2, \dots, 6)$ are constants.

By substituting (33) into the boundary conditions (23), we obtain:

$$\sum_{i=1}^{6} c_i \{ \varsigma^i, \beta^i, \gamma^i \}^T E^{(i)} = 0, \ \ (r = a, b), \tag{34}$$

where

$$\varsigma^i = -F_0{}^{(i)} + r\, s^{(i)}\, G_0{}^{(i)} + (-F_1{}^{(i)} - G_0{}^{(i)} + r\, s^{(i)} G_1{}^{(i)} + r\,(G_0{}^{(i)})'\,)/m + (-F_2{}^{(i)} - G_1{}^{(i)} + \ r\, s^{(i)} G_2{}^{(i)} + r(G_1{}^{(i)})')/m^2 + \cdots,$$

Power System Technology
ISSN:1000-3673
Accepted: 07-03-2024 Revised: 12-02-2024 Received: 16-01-2024

$$\beta^{(i)} = (1 + r^2(s^{(i)})^2 F_0{}^{(i)} + ((r\, s^{(i)} + r^2(s^{(i)})')F_0{}^{(i)} + 2r^2 s^{(i)}(F_0{}^{(i)})' + (1 + r^2(s^{(i)})^2)F_1{}^{(i)})/m + ((-1 + r^2\alpha^2)F_0{}^{(i)} + r(F_0{}^{(i)})' + r^2(F_0{}^{(i)})'' + (r\, s^{(i)} + r^2(s^{(i)})' F_1{}^{(i)} + 2r^2 s^{(i)}(F_1{}^{(i)})' + \left(1 + r^2(s^{(i)})^2\right) F_2{}^{(i)})/m^2 + \cdots,$$

$$\gamma^{(i)} = 2s^{(i)}(A^2 - (a^2 - r^2)\lambda_z)F_0{}^{(i)} - r^2\lambda_z{}^2 K_0{}^{(i)} + (2s^{(i)}(A^2 - (a^2 - r^2)\lambda_z)F_1{}^{(i)} + 2(A^2 - (a^2 - r^2)\lambda_z)(F_0{}^{(i)})' - r^2\lambda_z{}^2 K_1{}^{(i)})/m + (2s^{(i)}(A^2 - (a^2 - r^2)\lambda_z) F_2{}^{(i)} + 2(A^2 - (a^2 - r^2)\lambda_z)(F_1{}^{(i)})' - r^2\lambda_z{}^2 K_2{}^{(i)})/m^2 + \cdots.$$

The boundary conditions (34) yield a matrix equation of the form:

$$\sum_{j=1}^{6} M_{ij} c_j = 0, \ (i = 1, 2, \ldots 6), \tag{35}$$

where, c_j $(j = 1, 2, \ldots\ 6)$ are constants and

$$M_{1j} = \varsigma^{(j)}(a), M_{2j} = \beta^{(j)}(a), M_{3j} = \gamma^{(j)}(a),$$
$$M_{4j} = E^{(j)}(b)\varsigma^{(j)}(b), M_{5j} = E^{(j)}(b)\beta^{(j)}(b), M_{6j} = E^{(j)}(b)\gamma^{(j)}(b).$$

5 Analyse the outer layer eigenvalues

The expansion of equation (35) consists of terms proportional to $E^{(j)}$, $(j = 1, 2, \ldots, 6)$. For $A - 1 = O(1)$, we have $b - a = O(1)$, so $E^{(1)}$, $E^{(3)}$ and $E^{(5)}$ are exponentially large, whereas $E^{(2)}, E^{(4)}$ and $E^{(6)}$ are exponentially small. Hence, we have:

$$\frac{\det(M_{ij})}{E^{(1)}E^{(3)}E^{(5)}} = \begin{bmatrix} \varsigma^{(1)}(b) & \varsigma^{(3)}(b) & \varsigma^{(5)}(b) \\ \beta^{(1)}(b) & \beta^{(3)}(b) & \beta^{(5)}(b) \\ \gamma^{(1)}(b) & \gamma^{(3)}(b) & \gamma^{(5)}(b) \end{bmatrix} \begin{bmatrix} \varsigma^{(2)}(a) & \varsigma^{(4)}(a) & \varsigma^{(6)}(a) \\ \beta^{(2)}(a) & \beta^{(4)}(a) & \beta^{(6)}(a) \\ \gamma^{(2)}(a) & \gamma^{(4)}(a) & \gamma^{(6)}(a) \end{bmatrix} + \text{EST},$$

where EST stands for the exponentially small terms. The matrix equation (35) can be replaced, with an error that is exponentially small, by the two matrix equations:

$$\begin{bmatrix} \varsigma^{(1)}(b) & \varsigma^{(3)}(b) & \varsigma^{(5)}(b) \\ \beta^{(1)}(b) & \beta^{(3)}(b) & \beta^{(5)}(b) \\ \gamma^{(1)}(b) & \gamma^{(3)}(b) & \gamma^{(5)}(b) \end{bmatrix} \begin{Bmatrix} c_1 \\ c_3 \\ c_5 \end{Bmatrix} = 0,$$

and

$$\begin{bmatrix} \varsigma^{(2)}(a) & \varsigma^{(4)}(a) & \varsigma^{(6)}(a) \\ \beta^{(2)}(a) & \beta^{(4)}(a) & \beta^{(6)}(a) \\ \gamma^{(2)}(a) & \gamma^{(4)}(a) & \gamma^{(6)}(a) \end{bmatrix} \begin{Bmatrix} c_2 \\ c_4 \\ c_6 \end{Bmatrix} = 0.$$

To obtain the non-trivial solution, we should have:

$$\begin{vmatrix} \varsigma^{(2)}(a) & \varsigma^{(4)}(a) & \varsigma^{(6)}(a) \\ \beta^{(2)}(a) & \beta^{(4)}(a) & \beta^{(6)}(a) \\ \gamma^{(2)}(a) & \gamma^{(4)}(a) & \gamma^{(6)}(a) \end{vmatrix} = 0. \tag{36}$$

With the help of functions $s^{(i)}(r)$, $F_0{}^{(i)}, F_1{}^{(i)}, F_2{}^{(i)}, (i = 1, 2, \ldots, 6)$, and by substituting $a = \lambda_a A$ and due to the expansion of (36) of course with the aid of Mathematica for different $\lambda_z = (1, 2, 3, 4)$ and $L = 5$, we obtain:

$\lambda_a = 0.543689 + (\ 0.352201 + 1.30374 \times 10^{-19} \times A^2\ )/m$, (for $\lambda_z = 1,\ L = 5$)

Power System Technology
ISSN:1000-3673
Accepted: 07-03-2024 Revised: 12-02-2024 Received: 16-01-2024

$\lambda_a = 0.384446 + (0.249044 + 2.37471 \times 10^{-21} \times A^2)/m$, (for $\lambda_z = 2,\ L = 5$)
$\lambda_a = 0.313899 + (0.203343 + 1.04439 \times 10^{-18} \times A^2)/m$, (for $\lambda_z = 3,\ L = 5$)
$\lambda_a = 0.271845 + (0.176101 + 1.28674 \times 10^{-19} \times A^2)/m$. (for $\lambda_z = 4,\ L = 5$)

By a similar procedure, we can obtain λ_a for $L = 5, 10$ and different λ_z $(= 1,\ 2,\ 3,\ 4,\ 5)$. By comparing the outer layer asymptotic results with the numerical data, we easily observe that the results obtained by both methods are coincident of course for the small values of A/B, but the asymptotic results fail to approximate the numerical data for A/B close to unity. The eigenvalues λ_a and λ_b corresponding to A/B close to unity are discussed in the next section.

6 The eigenvalues of the inner layer

Since for small $A-1$ (i.e. A close to unity), $b-a$ is small and $E^{(1)}$, $E^{(3)}$ and $E^{(5)}$ aren't exponentially large and hence, $\det(M_{ij})/(E^{(1)}E^{(3)}E^{(5)})$ fail to approximate $\det(M_{ij})$, therefore $E^{(1)}$, $E^{(3)}$ and $E^{(5)}$ become $O(1)$ and since $A-1 = O(1/m)$, all the M_{ij}'s become $O(1)$; so it is necessary to write:

$$A = 1 + \zeta/m, \tag{37}$$

where, ζ is an $O(1)$ constant and try to find an asymptotic expansion for λ_a in terms of η_i $(i = 1, 2, \dots)$ as follows:

$$\lambda_a = \eta_1 + \eta_2/m + \eta_3/m^2 + \cdots, \tag{38}$$

where, $\eta_1,\ \eta_2,\ \eta_3, \dots$, are to be determined. On substituting the expansions of (37) and (38) into $\lambda_a = a/A$ and $\lambda_b = b/B$, we obtain:

$$a = \eta_1 + (\zeta\,\eta_1 + \eta_2)/m + (\zeta\,\eta_2 + \eta_3)/m^2 + \cdots,$$
$$b = \eta_1 + (-\zeta + \zeta\,\eta_1{}^2\lambda_z + \eta_1\eta_{2\,\lambda_z})/(m\,\eta_1\lambda_z) + \cdots.$$

By substituting a into (36) and by equating the coefficients of the like powers of m by Mathematica, to the leading order, det (M_{ij}) yields:

$$8\,\eta_1{}^2\,\lambda_z(1 + \eta_1{}^4\,\lambda_z{}^2)^2 - 8\,\eta_1{}^2\lambda_z\,cosh[\zeta]\,cosh\left[\frac{\zeta}{\eta_1{}^2\,\lambda_z}\right]\left(1 + \eta_1{}^4\,\lambda_z{}^2\right)^2 +$$
$$sinh[\zeta]sinh\left[\frac{\zeta}{\eta_1{}^2\lambda_z}\right](1 + 20\,\eta_1{}^4\lambda_z{}^2 + 6\,\eta_1{}^8\,\lambda_z{}^4 + 4\,\eta_1{}^{12}\lambda_z{}^6 + \eta_1{}^{16}\,\lambda_z{}^8) = 0. \tag{39}$$

By solving (39) we obtain η_1. Similarly, we can easily obtain the equations satisfied by η_2 and η_3, by considering the next-order terms, but they are too long to be written out here. For any given mode number m and any value of A close to unity, the corresponding value of ζ is determined by (37). In view of equations satisfied by $\eta_1,\ \eta_2,\ \eta_3$ and the relation (38), the value of λ_a is determined.

7 Results

The curves of λ_a and λ_b (the eigenvalues) against A/B for different values of λ_z and m have been displayed respectively in Fig. 1 and Fig. 2. We observe that the dependency of λ_a and λ_b on A/B have a boundary layer structure such that, the asymptotic results of the outer layer for any values of A/B and the asymptotic inner layer data corresponding to equation (38) for A/B close to unity are coincident with the obtained numerical results. Due to increasing the axial stretch both of the inner and outer radiuses decrease by increasing λ_z (see Fig. 1 and Fig. 2). Now we deal with the variations of the inner radius λ_a with respect to A/B. By analyzing the

Accepted: 07-03-2024 *Revised: 12-02-2024* *Received: 16-01-2024*

plotted curves, we conclude that for the specific A/B i.e. 0.85 for $m = 20$ and near 0.8 for m = 10, the inner radius changes uniformly due to axial stretch. But for the thinner cylinders (as A/B is close to unity), the effect of the inner pressure is more than the effect of the axial stretch and this is the case of increasing the inner radius of the cylinder. Also, both of the results i.e. asymptotic and numerical ones obtained for the thicker shells of course for the higher mode numbers (m), are also more coincident (i.e. as the cylinder gets thicker the results of the two methods are getting closer). We noticed from the Fig. 2 that, as the cylinder gets thinner (of course for a certain amount of A/B), the effects of the axial stretch are more than the inner pressure applied to the cylinder, so in such a case, the outer radius decreases, but for the cylinders with very thin layers, the effects of the inner pressure increase and it is clear that the effect of the axial stretch decreases, (this is true for different cases of Fig. 2). What we described for λ_a will appear similarly for λ_b of course for A/B close to unity.

8 Conclusion

In this research, the eigenvalues result from the asymmetric bifurcation of the neo-Hookean incompressible cylindrical shell obtained by using the WKB method. The most topics results from this research are as follows:

- The asymptotic and numerical results become closer, as the mode number increases;
- The asymptotic results are completely coincident with the counterpart numerical data for A/B close to unity;
- The rate of change of the outer radius to a certain A/B is fixed, and as the shell is becoming thinner, the rate of change of the inner radius is increasing;
- The change rate of the outer radius decreases first, and then for the thinner shells, the rate of change of the outer radius increases;
- By solving the eigenvalue problem by using the asymptotic WKB method to the leading order six independent solutions for S(r) are obtained, where two roots are repeated and the other two are not. For the eigenvalue problem to be solvable, corresponding solvability conditions related to the mentioned roots are first-order differential equations;

References

[1] N. Abdolalian and M. Sanjaranipour, Evaluation of the buckling modes of a rotating neo-Hookean cylinder by WKB method, *Iranian Journal of Science and Tecknology, Transactions A: Science*. **42**(3) (2018) 1383-1393.

[2] C. D. Coman and M. Destrade, Some Asymptotic Results for Bifurcations in Pure Bending of Rubber Block, *Quarterly Journal of Mechanics and Applied Mathematics*. **61**(3) (2008) 395-414.

[3] Y. B. Fu, Some Asymptotic Results Concerning the Buckling of a Spherical Shell of arbitrary Thickness, *International Journal of Non-Linear Mechanics*. **33** (6) (1998) 1111-1122.

[4] Y. B. Fu and Y. P. Lin, A WKB analysis of the buckling of an everted neo-Hookean cylindrical tube, *Mathematics and Mechanics of Solids*. **7**(5) (2002) 483-501.

Power System Technology
ISSN:1000-3673
Accepted: 07-03-2024 Revised: 12-02-2024 Received: 16-01-2024
[5] Y. B. Fu and M. Sanjaranipour, WKB method with repeated roots and its application to the buckling analysis of an everted cylindrical tube, *Siam Journal on Applied Mathematics*. **62**(6) (2002) 1856-1871.
[6] A. E. Green and A. J. M. Spencer, The stability of a circular cylinder under finite extension and torsion. *Studies in Applied Mathematics*, **37**(1-4) (1959) 316-338.
[7] V. Z. Gristchak, D. D. Hryshchak and N. M. Dyachenko, Efficient approximate analytic solution for the problem of stability of a three-layer conic shell under combined loading, *Journal of Mathematical Sciences*. **254**(1) (2021) 71-88.
[8] D. M. Haughton and Y. C. Chen, Asymptotic bifurcation results for the eversion of elastic shells, *Zeitschrift fur Angewandte Mathematik und Physik*. **54**(2) (2003) 191-212.
[9] D. M. Haughton and R. W. Ogden, Bifurcation of an inflated circular cylinder of Elastic material under axial loading --- I. Membrane theory for thin-walled tubes, *Journal of the Mechanics and Physics of Solids*, **27**(3) (1979) 179-212.
[10] D. M. Haughton and R. W. Ogden, Bifurcation of an inflated circular cylinder of Elastic material under axial loading ---II. Exact theory for thick-walled tubes, *Journal of the Mechanics and Physics of Solids*. **27**(5-6) (1979) 489-512.
[11] D. M. Haughton and A. Orr, On the eversion of incompressible elastic cylinder, *International Journal of Non-Linear Mechanics.* **30**(2) (1995) 81-95.
[12] Y. Liu, Axial and circumferential buckling of a hyper-elastic tube under restricted compression, International Journal of Non-Linear Mechanics, **98** (2018) 145-153.
[13] M. Metcalf, J. Reid, and M. Cohen, *Modern Fortran explained*, Oxford University, New York, 2011.

[14] E. Omugbe, O. E. Osafile and M. C. Onyeaju, Mass Spectrum of Mesons via the WKB approximation method. Advances in High Energy Physics, (2020).

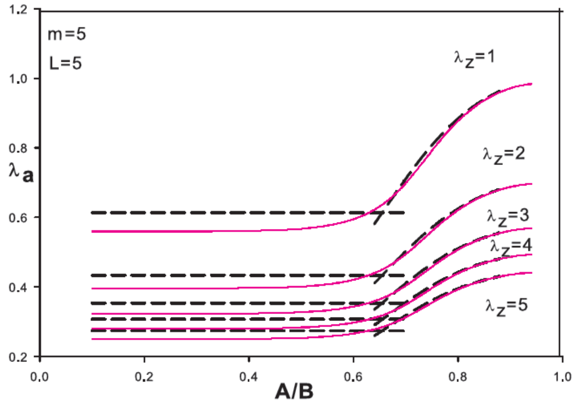
[15] S. A. Sahu, S. Nirwal, An asymptotic approximation of Love wave frequency in a piezo-composite structure: WKB approach, *Waves in Random and Complex Media*. **31**(1) (2021) 117-145.
[16] M. Sanjaranipour, A WKB analysis of the buckling condition for a cylindrical shell of arbitrary thickness subjected to an external pressure, *IMA Journal of Applied Mathematics*. **70**(1) (2005) 147-161.
[17] M. Sanjaranipour, WKB Analysis of the Buckling of a Neo-Hookean Cylindrical Shell of Arbitrary Thickness Subject to an External pressure, International Journal of applied Mechanics, **2**(4) (2010) 857-870.
[18] M. Sanjaranipour and A. Irandegani, The axisymmetric bifurcation analysis of an elastic cylindrical shell subjected to external pressure and axial loading, *Iranian Journal of Science and Technology*. **38**(3. 1) (2014) 343-348.
[19] M. Sanjaranipour, A. Hatami and N. Abdolalian, Another Approach of WKB Method for the Stability Analysis of the Bending of an Elastic Rubber Block, *International Journal of Engineering Science*. **62** (2013) 1-8.

Power System Technology
ISSN:1000-3673
Accepted: 07-03-2024 *Revised: 12-02-2024* *Received: 16-01-2024*

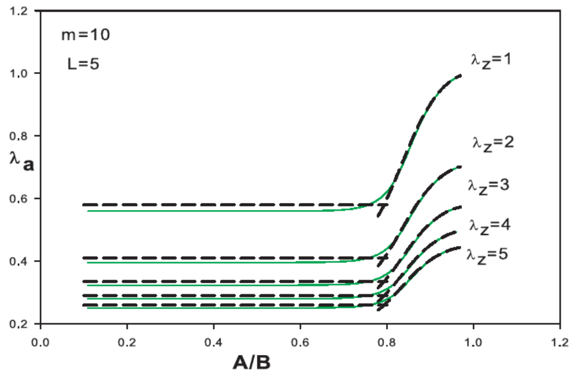
[20] R. L. Sierakowski, C. T. Sun and I. K. Ebcioglu, Instability of a hollow rubber-like cylinder under initial stress, *International Journal of Engineering Science*. **10**(12) (1975) 1311-1324.

[21] E. W. Wilkes, On the stability of a circular tube under end thrust, *Quarterly Journal of Mechanics and Applied Mathematics*, **8**(1) (1955) 88-100.

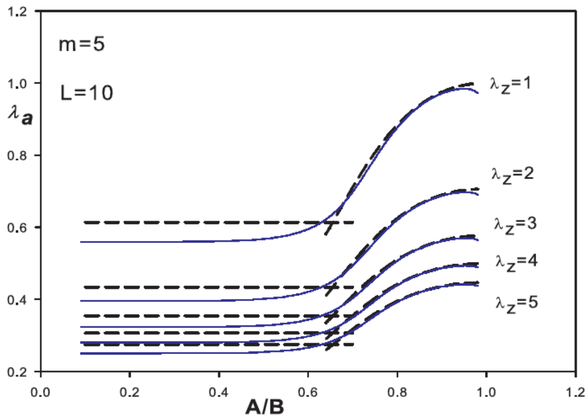
[22] Y. Zhu, X. Y. Luo and R. W. Ogden, Asymmetric bifurcation of thick-walled circular cylindrical elastic tubes under axial loading and external pressure, *International journal of solids and structures*, **45**(11-12) (2008) 3410-3429.



(a)



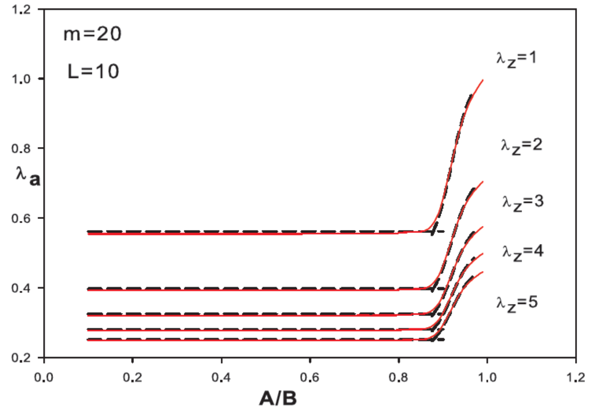
(b)



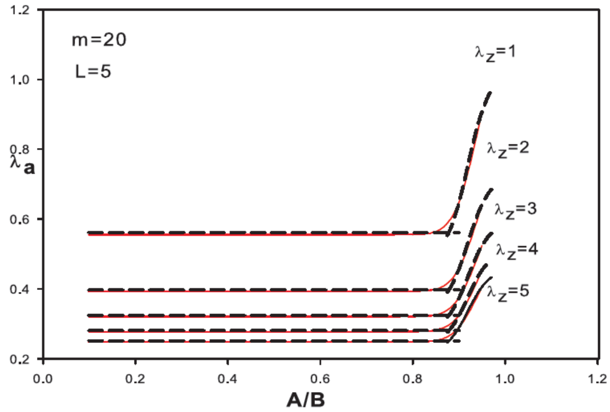
(c)

Power System Technology
ISSN:1000-3673

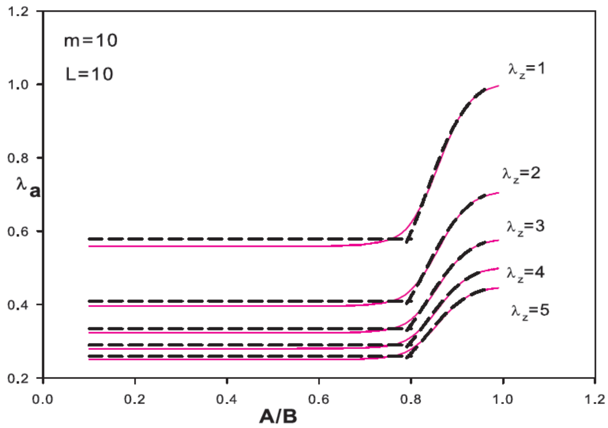
Accepted: 07-03-2024 *Revised: 12-02-2024* *Received: 16-01-2024*



(d)



(e)

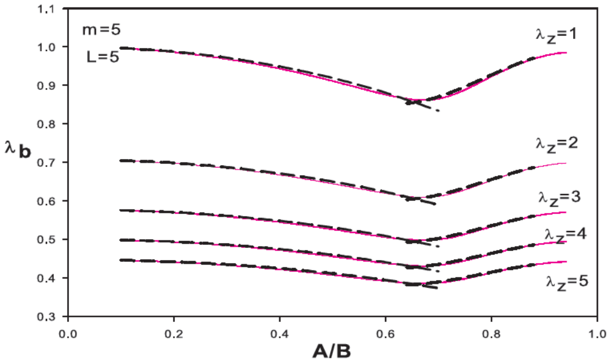


(f)

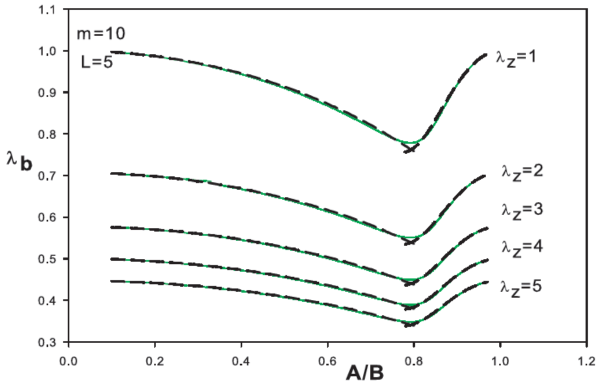
Fig. 1 The curves of λ_a against A/B for $\lambda_z = 1, 2, \dots, 5$ and $m = 5, 10, 20$ and $L = 5, 10$. Dashed lines: asymptotic results (outer and inner layers); solid lines: Compound Matrix results.

Power System Technology
ISSN:1000-3673

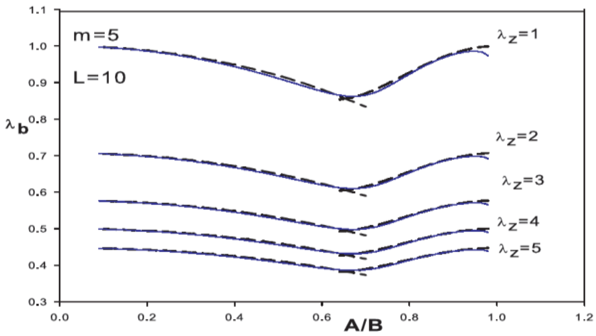
Accepted: 07-03-2024 *Revised: 12-02-2024* *Received: 16-01-2024*



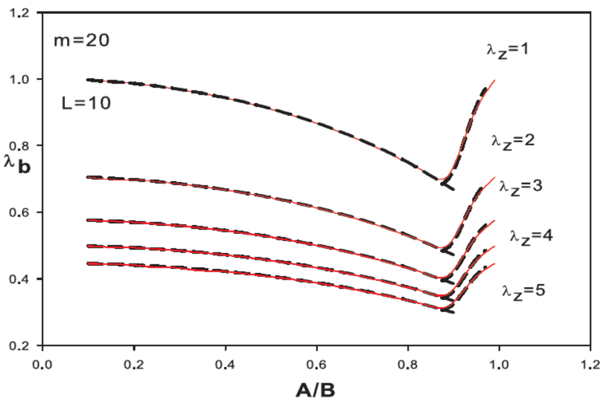
(a)



(b)



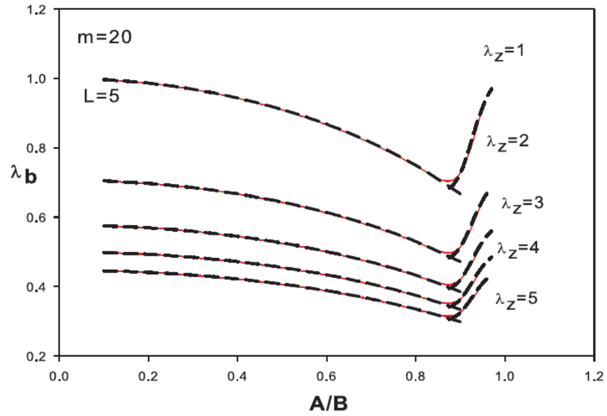
(c)



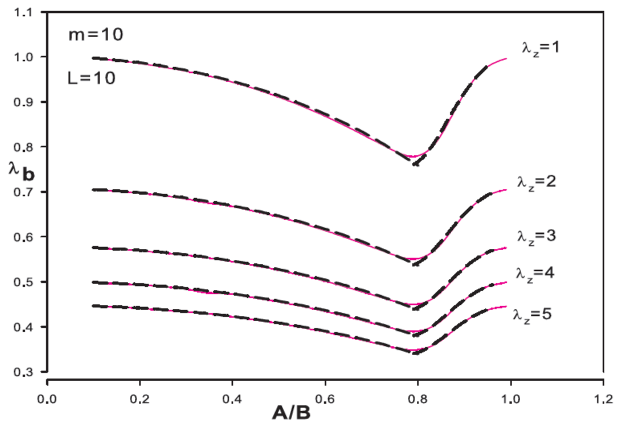
(d)

Power System Technology
ISSN:1000-3673

Accepted: 07-03-2024 *Revised: 12-02-2024* *Received: 16-01-2024*



(e)



(f)

Fig. 2 The curves of λ_b against A/B for $\lambda_z = 1, 2, \ldots, 5$ and $m = 5, 10, 20$ and $L = 5, 10$. Dashed lines: asymptotic results (outer and inner layers); solid lines: Compound Matrix results.