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On the Construction of a Strong Fuzzy Resolving Set of Fuzzy Graphs

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Abstract—Assume $\tilde{G}(\tilde{V},\tilde{E})$ is a fuzzy graph with fuzziness on vertex set $\tilde{V}$ and edge set $\tilde{E}$. Let $\tilde{V}$ be a fuzzy set on V with $|V| = n$ and $\tilde{H} \subseteq \tilde{V}$. The representation of $\tilde{V} - \tilde{H}$ concerning $\tilde{H}$ is an ordered l–tuples $h_i/\tilde{H} = (w(v_iv_1), w(v_iv_2), \cdots, w(v_iv_l))$, where $h_i = (v_i, \sigma(v_i)) \in \tilde{V} - \tilde{H}$ for $i = l+1, l+2, \cdots, n$ and $w(v_iv_j) = \mu^{\infty}(v_iv_j)$ indicates the maximum strength of all paths connecting u and v in $\tilde{G}$ for $1 \leq j \leq l$. The "fuzzy resolving set (FRS)" $\tilde{H}$ of $\tilde{G}$ is the subset of $\tilde{V}$ in which the representations of $\tilde{V} - \tilde{H}$ concerning $\tilde{H}$ are all distinct. The different representations of $\tilde{V} - \tilde{H}$ are row-matrices that could be linearly independent or linearly dependent on each other. This paper proposes the "strong fuzzy resolving set (SFRS)" of fuzzy graphs, i.e., the FRS of $\tilde{G}$ in which the representations of $\tilde{V} - \tilde{H}$ concerning $\tilde{H}$ are linearly independent. The strong resolving number (SRN) is the least cardinality of the underlying set of the SFRS. This article also discusses the connection that exists between an SFRS and the FRS of fuzzy graphs. Furthermore, it is proved that the fuzzy labeling graph $\tilde{G}$ of n vertices, whose the underlying graph is a cycle C_n has the SRN $\left\lceil \frac{n}{2} \right\rceil$.

Keywords—Fuzzy graph, Fuzzy resolving set, Linearly independent, Strong fuzzy resolving set, Strong resolving number.

1. INTRODUCTION

Zadeh proposed a "fuzzy set" in 1965. Ten years later, Rosenfeld defined fuzzy graphs in 1975 [1]. He has defined almost all the basic definitions in fuzzy graph theory, such as fuzzy path, connectivity, strength of the path, strongest path, fuzzy tree, cut nodes, bridge, etc. Fuzzy graphs can be applied to many real-life problems, which include cluster analysis [2], traffic networks ([3],[4]), telecommunications [5], time detraction of sign image segmentation [6], assignment of frequency in radio stations [7], coloring the world political map to show the

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strength of relationships between the countries [8], job oriented web sites [9], disaster management system [10], and optimization in business strategy [11].

Slater et al. proposed the "resolving set (RS)" of graphs [12]. Slater called it a "locating set" and a "locating number", whereas Harary and Melter introduced the "metric dimension" [13]. Given a graph $G(V, E)$ and $H \subseteq V$, the representation of $v \in V$ concerning H is a vector $(d_G(v, h))$ for each $h \in H$, where $d_G(v, h)$ denotes the distance between v and h in G. The subset H is mentioned as the RS of G if each vertex in G has a different representation of $V - H$ concerning H [15]. The least cardinality of the RS is called the "resolving number (RN)" of G. One of the applications of the RS and the RN of a graph is to determine the locations and the minimal number of fire sensors in a building [14]. However, there are indeterminate phenomena related to the fire sensor installation, such as temperature, weather, the types of sensors, safety level, etc. Thus, we need a framework with a fuzzy graph structure to handle the problem of fire sensor installation.

Jiny and Shanmugapriya introduced the "fuzzy resolving number" in 2019 [16]. They also came up with the modified "fuzzy resolving number" [17], the "fuzzy super resolving number" [18], the characteristic of the FRS [19], the RN of fuzzy digraphs [20], and the "regular FRS" [21].

In a fuzzy graph $\tilde{G}(\tilde{V}, \tilde{E})$, the representation of $\tilde{V} - \tilde{H}$ concerning $\tilde{H}$ for any subset $\tilde{H} \subseteq \tilde{V}$ is a row-matrix whose elements lie between 0 and 1. Two row-matrices are called linearly independent if one cannot be represented as a linear combination of the other. In certain cases, these row-matrices are linearly independent. Sometimes, we need two row-matrices that are linearly independent for application purposes. For example, we need strong positions to locate the fire sensors in a building to optimize the minimal number of sensors that will be used. This encourages us to propose the "strong fuzzy resolving set (SFRS)" and the "strong resolving number (SRN)" of fuzzy graphs in this paper. We also examine the characteristics of the SFRS as theorems and corollaries. Moreover, we investigate the connection between the SFRS and the FRS of fuzzy graphs.

The organization of this article is the following: preliminaries are given in Section 2, and Section 3 looks at the SFRS and associated characteristics. The conclusion and suggestions for further study are provided in Section 4.
For the sake of simplicity, we summarize some abbreviations in Table 1.

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TABLE I
SOME ESSENTIAL ABBREVIATIONS

Abbreviation	Meaning
RS	"Resolving set"
RN	"Resolving number"
FRS	"Fuzzy resolving set"
SFRS	"Strong fuzzy resolving set"
SRN	"Strong resolving number"
MF	"Membership function"
FG	"Fuzzy graph"

2. PRELIMINARIES

Some fundamental ideas that are necessary for this study are outlined in this section.

Definition 2.1. [22] *Given a universal set* X. *Let* $\tilde{A} = \{(a, \mu_{\tilde{A}}(a)) | a \in X\}$ *and* $\tilde{B} = \{(b, \mu_{\tilde{B}}(b)) | b \in X\}$ *be two fuzzy sets on* X. *The difference between* $\tilde{A}$ *and* $\tilde{B}$ *is a fuzzy set* $\tilde{A} - \tilde{B}$ *with* $\mu_{\tilde{A}-\tilde{B}}(v) = \min\{\mu_{\tilde{A}}(v), 1 - \mu_{\tilde{B}}(v) | v \in X\}$.

Definition 2.2. [22] *A graph* $\tilde{G}(\tilde{V}, \tilde{E})$ *is called a fuzzy graph (FG) if* $\tilde{V}$ *is the fuzzy vertex set on* V *(non-empty) with a membership function (MF)* $\sigma: V \to [0, 1]$ *and* $\tilde{E}$ *is the fuzzy edge set with an MF* μ *from* $E \subseteq V \times V$ *to* $[0, 1]$ *in such a way* $\mu(u_1u_2) \le \sigma(u_1) \wedge \sigma(u_2)$ *for all* $u_1, u_2 \in V$. *We can also express* $\tilde{G}(\tilde{V}, \tilde{E})$ *as* $\tilde{G}(V, \sigma, \mu)$.
Further, an underlying graph of the FG $\tilde{G}$ *is a crisp graph* $G^*(V^*, E^*)$, *where* $V^* = \{x \in V: \sigma(x) > 0\}$ *and* $E^* = \{xy \in V \times V: \mu(xy) > 0\}$.

Definition 2.3. [23] *Let* $\tilde{V}$ *be a fuzzy set on* V. *A fuzzy labeling graph is an FG* $\tilde{G}(\tilde{V}, \tilde{E})$ *which has bijective membership functions* σ *and* μ *in such a way that the edges and vertices have dissimilar membership degrees and* $\mu(u_1u_2) \le \sigma(u_1) \wedge \sigma(u_2)$ *for all* $u_1, u_2 \in V$.

Definition 2.4. [20] *Two row-matrices are said to be linearly independent if one cannot be stated as linear combination of the other.*

Definition 2.5. [22] *A sequence of distinct vertices* $P = v_1, v_2, \cdots, v_n$ $(n \ge 2)$ *in FG* $\tilde{G}(\tilde{V}, \tilde{E})$ *such that* $\mu(v_jv_{j+1}) > 0$ *for* $j = 1,2, \cdots, n-1$ *is called a path* P. *The weight or strength of* P *is the minimum of* $\mu(v_jv_{j+1})$. *The maximum strength of all paths connecting* u *and* v *in* $\tilde{G}$ *is named the strength of connectedness between* u *and* v, *denoted as* $\mu^{\infty}(uv)$.

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Definition 2.6. [17] *Given a fuzzy set $\widetilde{V}$ on V and FG $\tilde{G}(\widetilde{V},\tilde{E})$ with ($|V| \geq 3$). The FRS $\widetilde{H}$ is the subset $\widetilde{H} \subseteq \widetilde{V}$ with the cardinality $|H^*| \geq 2$ such that the representations of $\widetilde{V}-\widetilde{H}$ concerning $\widetilde{H}$ are all distinct.*
If $\widetilde{V}=\left\{\left(v_{1,}\sigma(v_1)\right),\left(v_{2,}\sigma(v_2)\right),\left(v_{3,}\sigma(v_3)\right),\cdots,\left(v_{n,}\sigma(v_n)\right)\right\}$ and $\widetilde{H}=\{(v_1,\sigma_{\widetilde{H}}(v_1)),$ $(v_2,\sigma_{\widetilde{H}}(v_2)),\cdots,(v_l,\sigma_{\widetilde{H}}(v_l))\}$ where $l<n$, then the representation of $\widetilde{V}-\widetilde{H}$ concerning $\widetilde{H}$ is an ordered $l-$tuples:

$$h_i/\widetilde{H}=(w(v_iv_1),w(v_iv_2),\cdots,w(v_iv_l))$$

where $h_i=\left(v_i,\sigma(v_i)\right)\in\widetilde{V}-\widetilde{H}$ for $i=l+1,l+2,\cdots,n$ and $w\left(v_iv_j\right)=\mu^{\infty}(v_iv_j)$ for $1 \leq j \leq l$. The least cardinality of the underlying set of the FRS is mentioned as the RN of fuzzy graph $\tilde{G}$, symbolized as $Fr(\tilde{G})$.

According to Definition 2.6, the FG $\tilde{G}$ with n vertices has the RN $Fr(\tilde{G}) \leq n-1$.

Example 2.1. Determine the FRS and the RN of the FG $\tilde{G}$ in Fig.1.

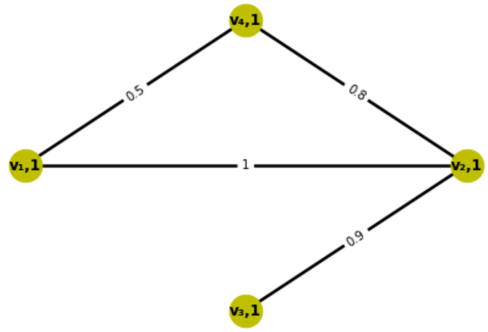

Fig. 1. The FG $\tilde{G}(\widetilde{V},\tilde{E})$ for Example 2.1

Let $V=v_1,v_2,\cdots,v_n$ and $\sigma(v_j)=1$ for $j=1,2,\cdots,n.$ Meanwhile, the edges in $\tilde{E}$ have membership degrees as follows:

$$\mu=\begin{bmatrix} & v_1 & v_2 & v_3 & v_4\\ v_1 & 0 & 1 & 0 & 0.5\\ v_2 & 1 & 0.9 & 0 & 0.8\\ v_3 & 0 & 0.9 & 0 & 0\\ v_4 & 0.5 & 0.8 & 0 & 0\end{bmatrix}.$$

Let us take $\widetilde{H}=\{h1=(v_1,1),h2=(v_2,1)\}\}$. Then, $\widetilde{V}-\widetilde{H}=\{h_3=(v_3,1),h_4=(v_4,1)\}$.
We get:

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$$h_3/\widetilde{H} = \left(w(v_3v_1), w(v_3v_2)\right) = (0.9,0.9)$$
$$h_4/\widetilde{H} = \left(w(v_4v_1), w(v_4v_2)\right) = (0.8,0.8).$$

All of the representations of $\widetilde{V} - \widetilde{H}$ concerning $\widetilde{H}$ are different. Thus, $\widetilde{H}$ is the FRS of $\tilde{G}$ and $F_r(\tilde{G}) = 2$.

3. Strong Fuzzy Resolving Set

When $\widetilde{H}$ is the FRS of the FG $\tilde{G}$, the representations of $\widetilde{V} - \widetilde{H}$ concerning $\widetilde{H}$ are different. However, if we consider each representation as a row-matrix, then two row-matrices can be linearly independent or linearly dependent. For example, if the representations are (0.2, 0.2) and (0.4, 0.4), then they are linearly dependent. This motivates us to define a strong condition for the FRS in Definition 3.1.

Definition 3.1. *Consider the FG $\tilde{G}(\widetilde{V}, \tilde{E})$ and $\widetilde{H} \subseteq \widetilde{V}$ with $|H^*| \geq 2$. Each representation of $\widetilde{V} - \widetilde{H}$ concerning $\widetilde{H}$ is a row-matrix. If all of the row-matrices are linearly independent of each other, then the subset $\widetilde{H}$ is mentioned as the SFRS of $\tilde{G}$. The minimum SFRS is an SFRS with the least cardinality of its underlying set. The SRN of G is written as $F_{sr}(\tilde{G})$, and it refers to the cardinality of the minimum SFRS.*

Theorem 3.1. *If $\widetilde{H}$ is an SFRS of the FG $\tilde{G}$, then it is an FRS of $\tilde{G}$.*

Proof. Given FG $\tilde{G}(\widetilde{V}, \tilde{E})$. Let $\widetilde{H} \subseteq \widetilde{V}$ be an SFRS. It is obvious that the representations of $\widetilde{V} - \widetilde{H}$ regarding $\widetilde{H}$ are row-matrices, and they are linearly independent of each other. Suppose that any two row-matrices are equal, then they are linearly dependent on each other. This statement appears to contradict the claim that $\widetilde{H}$ is an SFRS of $\tilde{G}$. Hence, no two row-matrices are equal. It means that all of the representations of $\widetilde{V} - \widetilde{H}$ concerning $\widetilde{H}$ are distinct. Thus, every SFRS of the FG $\tilde{G}$ is an FRS of $\tilde{G}$. ■

Corollary 3.1. *Each FRS of the FG $\tilde{G}(\widetilde{V}, \tilde{E})$ is not always an SFRS of $\tilde{G}$.*

Proof. If $\widetilde{H}$ is the FRS of a fuzzy graph $\tilde{G}$, then the representations of $\widetilde{V} - \widetilde{H}$ concerning $\widetilde{H}$ are distinct. However, this may not imply that the representations must be linearly independent. Hence, an FRS of $\tilde{G}$ need not be an SFRS of $\tilde{G}$. ■

Theorem 3.2. *For the FG $\tilde{G}(\widetilde{V}, \tilde{E})$ with n vertices: $F_{sr}(\tilde{G}) \geq \left\lceil \frac{n}{2} \right\rceil$.*

Proof. Suppose that there is an SFRS $\widetilde{H} \subseteq \widetilde{V}$ where $|H^*| < \left\lceil \frac{n}{2} \right\rceil$.

Case 1: n is even.

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When n is even, the value $\left\lceil\frac{n}{2}\right\rceil = \frac{n}{2}$. Consider $|H^*| < \frac{n}{2}$, say $|H^*| = \frac{n}{2} - 1$. It implies that the set $\widetilde{V} - \widetilde{H}$ has $n - \frac{n}{2} + 1$ elements. That is $h_j/\widetilde{H}$ is a row matrix with $\left(\frac{n}{2} - 1\right)$ elements for each $h_j \in \widetilde{V} - \widetilde{H}$ with $j = 1,2,\cdots,\frac{n}{2} + 1$. There will be $\frac{n}{2} + 1$ such row matrices. Therefore, at least one pair of those will be linearly dependent where it contradicts to our assumption that $\widetilde{H}$ is an SFRS. Therefore, there does not exist an SFRS where the cardinality of its underlying set is less than $\left\lceil\frac{n}{2}\right\rceil$.

Case 2: n is odd.

When n is odd, we have $\left\lceil\frac{n}{2}\right\rceil = \frac{n+1}{2}$. Consider $|H^*| = \frac{n+1}{2} - 1 = \frac{n-1}{2}$. The set $\widetilde{V} - \widetilde{H}$ has $n - \frac{n-1}{2} = \frac{n+1}{2}$ elements. That is, $h_j/\widetilde{H}$ is a row matrix with $\left(\frac{n-1}{2}\right)$ elements for each $h_j \in \widetilde{V} - \widetilde{H}$ $j = 1,2,\cdots,\frac{n+1}{2}$. There will be $\frac{n+1}{2}$ such row matrices. Therefore, at least a pair of the elements will be linearly dependent. Which is a contradiction to our assumption. Hence, there does not exist an SFRS where its underlying set has cardinality less than $\left\lceil\frac{n}{2}\right\rceil$ if n is odd.

According to Case 1 and Case 2, the FG $\tilde{G}$ with n vertices has $F_{sr}(\tilde{G}) \geq \left\lceil\frac{n}{2}\right\rceil$. ■

Example 3.1. Determine the SFRS and the SRN of the FG $\tilde{G}$ in Fig. 2. Let $\widetilde{V} = \{(v_1, 1), (v_2, 0.9), (v_3, 1), (v_4, 0.9)\}$.

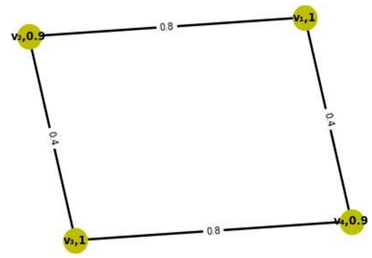

Fig. 2. Fuzzy graph $\tilde{G}(\widetilde{V}, \tilde{E})$ for Example 3.1

The set $\widetilde{V}$ has 6 subsets with two elements:

$$\widetilde{H}_1 = \{(v_1, 1), (v_2, 0.9)\};\ \widetilde{H}_2 = \{(v_1, 1), (v_3, 1)\};$$
$$\widetilde{H}_3 = \{(v_1, 1), (v_4, 0.9)\};\ \widetilde{H}_4 = \{(v_2, 0.9), (v_3, 1)\};$$
$$\widetilde{H}_5 = \{(v_2, 0.9), (v_4, 0.9)\};\ \widetilde{H}_6 = \{(v_2, 0.9), (v_4, 0.9)\}.$$

Now, $\widetilde{V} - \widetilde{H}_1 = \{h_3 = (v_3, 1), h_4 = (v_4, 0.9)\}$. We get

$$h_3/\widetilde{H}_1 = \left(\mu^\infty(v_3v_1), \mu^\infty(v_3v_2)\right) = (0.4,0.4),$$

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$$h_4/\widetilde{H}_1 = \left(\mu^{\infty}(v_4v_1), \mu^{\infty}(v_4v_2)\right) = (0.4,0.4).$$

As a consequence, $\widetilde{H}_1$ is not an SFRS of $\tilde{G}$.

Further, $\widetilde{V} - \widetilde{H}_2 = \{h_2 = (v_2, 0.9), h_4 = (v_4, 0.9)\}$. We have

$$h_2/\widetilde{H}_2 = \left(\mu^{\infty}(v_2v_1), \mu^{\infty}(v_2v_3)\right) = (0.8,0.4),$$

$$h_4/\widetilde{H}_2 = \left(\mu^{\infty}(v_4v_1), \mu^{\infty}(v_4v_3)\right) = (0.4,0.8).$$

It is clear that $\widetilde{H}_2$ is an FRS of $\tilde{G}$. Furthermore, all of the representations of $\widetilde{V} - \widetilde{H}_2$ concerning $\widetilde{H}_2$ are linearly independent of each other. Consequently, $\widetilde{H}_2$ is the SFRS of $\tilde{G}$. In the same way as the previous step, $\widetilde{H}_3, \widetilde{H}_4, \widetilde{H}_5$, and $\widetilde{H}_6$ are also the SFRS of $\tilde{G}$.

Hence, the underlying set of the SFRS of $\tilde{G}$ has the minimum cardinality of ``2''.

That is, $F_{sr}(\tilde{G}) = 2$. ■

Theorem 3.3. For any fuzzy graph $\tilde{G}$, the RN $F_r(\tilde{G}) \leq F_{sr}(\tilde{G})$.

Proof. Let $F_{sr}(\tilde{G}) = k$. Every SFRS of the FG $\tilde{G}$ is an FRS of $\tilde{G}$ (based on Theorem 3.1). Thus, $F_r(\tilde{G}) \leq k$. ■

In addition, we investigate the SFRS and SRN of fuzzy graphs with the cycles C_n as their underlying graphs for $n \geq 4$.

Theorem 3.4. Let $\widetilde{V}$ be a fuzzy set on V with $|V| = n$ for $n \geq 4$. We consider a fuzzy labeling graph $\tilde{G}(\widetilde{V}, \tilde{E})$ with the underlying graph in the form of the cycle C_n. We have $F_{sr}(\tilde{G}) = \left\lceil \frac{n}{2} \right\rceil$.

Proof. Given bijective functions $\sigma: V \to [0,1]$ such that the vertices in $\tilde{G}$ have different membership degrees and $\mu: E \to [0,1]$ in which all edges have distinct membership values.

Based on the definition of SFRS and the lower bound in Theorem 3.2: $\left\lceil \frac{n}{2} \right\rceil \leq F_{sr}(\tilde{G}) \leq n - 1$.

Let $\widetilde{V} = \{(x_i, \sigma(x_i)) | x_i \in V, i = 1,2, \cdots, n\}$.

The set $\widetilde{V}$ has $\binom{n}{\left\lceil \frac{n}{2} \right\rceil}$ subsets with $\left\lceil \frac{n}{2} \right\rceil$ elements as follows:

$$\widetilde{H}_{11} = \left\{(x_1, \sigma_{11}(x_1)), (x_2, \sigma_{11}(x_2)), \cdots, (x_{\left\lceil \frac{n}{2} \right\rceil}, \sigma_{11}\left(x_{\left\lceil \frac{n}{2} \right\rceil}\right))\right\},$$

…

$$\widetilde{H}_{1,\frac{1}{2}\binom{n}{\left\lceil \frac{n}{2} \right\rceil}+2} \left\{\left(x_1, \sigma_{1,\frac{1}{2}\binom{n}{\left\lceil \frac{n}{2} \right\rceil}+2}(x_1)\right), \cdots, \left(x_{n-1}, \sigma_{1,\frac{1}{2}\binom{n}{\left\lceil \frac{n}{2} \right\rceil}+2}(x_{n-1})\right), \left(x_n, \sigma_{1,\frac{1}{2}\binom{n}{\left\lceil \frac{n}{2} \right\rceil}+2}(x_n)\right)\right\},$$

if n is odd.

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$$\widetilde{H}_{1,\frac{1}{2}\binom{n}{\left\lceil\frac{n}{2}\right\rceil}} = \left\{\left(x_1, \sigma_{1,\frac{1}{2}\binom{n}{\left\lceil\frac{n}{2}\right\rceil}}(x_1)\right), \cdots, \left(x_{n-1}, \sigma_{1,\frac{1}{2}\binom{n}{\left\lceil\frac{n}{2}\right\rceil}}(x_{n-1})\right), \left(x_n, \sigma_{1,\frac{1}{2}\binom{n}{\left\lceil\frac{n}{2}\right\rceil}}(x_n)\right)\right\},$$

if n is even.

$$\widetilde{H}_{21} = \left\{\left(x_2, \sigma_{21}(x_2)\right), \left(x_3, \sigma_{21}(x_3)\right), \cdots, \left(x_{\left\lceil\frac{n}{2}\right\rceil+1}, \sigma_{21}\left(x_{\left\lceil\frac{n}{2}\right\rceil+1}\right)\right)\right\},$$

…

$$\widetilde{H}_{2p} = \left\{\left(x_2, \sigma_{2p}(x_2)\right), \cdots, \left(x_{n-1}, \sigma_{2p}(x_{n-1})\right), \left(x_n, \sigma_{2p}(x_n)\right)\right\}.$$

$$\widetilde{H}_{31} = \left\{\left(x_3, \sigma_{31}(x_3)\right), \left(x_4, \sigma_{31}(x_4)\right), \cdots, \left(x_{\left\lceil\frac{n}{2}\right\rceil+2}, \sigma_{31}\left(x_{\left\lceil\frac{n}{2}\right\rceil+2}\right)\right)\right\},$$

…

$$\widetilde{H}_{3q} = \left\{\left(x_3, \sigma_{3q}(x_3)\right), \cdots, \left(x_{n-1}, \sigma_{3q}(x_{n-1})\right), \left(x_n, \sigma_{3q}(x_n)\right)\right\}.$$

…

$$\widetilde{H}_{\binom{n}{\left\lceil\frac{n}{2}\right\rceil}} = \left\{\left(x_{n-\left\lceil\frac{n}{2}\right\rceil+1}, \sigma_{\binom{n}{\left\lceil\frac{n}{2}\right\rceil}}\left(x_{n-\left\lceil\frac{n}{2}\right\rceil+1}\right)\right), \left(x_{n-\left\lceil\frac{n}{2}\right\rceil+2}, \sigma_{\binom{n}{\left\lceil\frac{n}{2}\right\rceil}}\left(x_{n-\left\lceil\frac{n}{2}\right\rceil+2}\right)\right), \cdots, \left(x_n, \sigma_{\binom{n}{\left\lceil\frac{n}{2}\right\rceil}}(x_n)\right)\right\}.$$

Let us consider the subset $\widetilde{H}_{1t} = \{(x_{2t-1}, \sigma_{1t}(x_{2t-1}))\}$ for $t = 1,2,\cdots,\left\lceil\frac{n}{2}\right\rceil$.

We get $\widetilde{V} - \widetilde{H}_{1t} = \left\{\left(x_{2j}, \sigma(x_{2j})\right)\right\}$ where $j = 1,2,\cdots,s$ with $s = \begin{cases} \left\lceil\frac{n}{2}\right\rceil - 1, & \text{if n is odd} \\ \left\lceil\frac{n}{2}\right\rceil, & \text{if n is even.} \end{cases}$

The representations of $\widetilde{V} - \widetilde{H}_{1t}$ with respect to $\widetilde{H}_{1t}$ are as follows:

$$\left(\mu^\infty(x_2x_1), \mu^\infty(x_2x_3), \cdots, \mu^\infty(x_2x_{t'})\right),$$
$$\left(\mu^\infty(x_4x_1), \mu^\infty(x_4x_3), \cdots, \mu^\infty(x_4x_{t'})\right),$$
$$\cdots$$
$$\left(\mu^\infty(x_sx_1), \mu^\infty(x_sx_3), \cdots, \mu^\infty(x_sx_{t'})\right),$$

for $t' = 2\left\lceil\frac{n}{2}\right\rceil - 1$.

Since $\widetilde{G}$ is a fuzzy labeling graph and G^* is a cycle C_n there always exists a path connecting any pair of vertices with the strengths $\mu^\infty(x_1x_a)$ for $a = 2,3,\cdots,n$; $\mu^\infty(x_2x_b)$ for $b = 3,4,\cdots,n$; $\mu^\infty(x_3x_c)$ for $c = 4,5,\cdots,n$, and $\mu^\infty(x_{n-1}x_n)$ are all different. It ensures that all of the representations above are row-matrices that are linearly independent of each other. Hence, $\widetilde{H}_{1t}$ is an SFRS and $F_{sr}(\widetilde{G}) = \left\lceil\frac{n}{2}\right\rceil$. It completes the proof. ■

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4. Conclusion

This article discussed a new thought on the SFRS and the SRN (F_{sr}) of fuzzy graphs. We verified that each SFRS of fuzzy graphs was an FRS. The lower bound has been given, i.e., every FG $\tilde{G}$ with n vertices has $F_{sr}(\tilde{G}) \geq \left\lceil \frac{n}{2} \right\rceil$. Further, we proved that the $F_r(\tilde{G})$ was less than or equal to the $F_{sr}(\tilde{G})$. Hence, the SFRS was the more stronger condition than the FRS of fuzzy graphs. We also proved that the fuzzy labeling graph $\tilde{G}$ of n vertices, whose its underlying graph was cycle C_n, had $F_{sr}(\tilde{G}) = \left\lceil \frac{n}{2} \right\rceil$.

For future research, we can construct an algorithm to find the SFRS and SRN of fuzzy graphs. Moreover, we can determine the SRN of another class of fuzzy graphs according to the lower bound proven in this research.

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