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Exploring Fourth-Order Hankel Determinants for Subclasses of Analytic Function using the Lune Function

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Abstract:- In this paper, the aim was to investigate the Hankel fourth order determinants for the function like Convex and Starlike which was in the form of $(1 + z_1 \frac{f''(z_1)}{f'(z_1)})$ and $(z_1 \frac{f'(z_1)}{f(z_1)})$ respectively associated with the Lune operator. The basic goals of this paper were to investigate sharp coefficient bounds for convex and starlike function having normalized form $f(0) = 0$ and $f'(0) = 1$ of the function $f(z_1) = z_1 + b_2 {z_1}^2 + \ldots$ We derived several inequalities for the coefficients $|b_2|$ and $|b_3|$ of functions in these classes using a differential subordination method, as well as some known results on Hankel's determinant. Some existing results in the literature had been expanded and improved by our results. In order to demonstrate the clarity of our results, we provided some examples. The paper will be contributed to the theory and application of univalent functions in comprehensive analysis and geometry.

Keywords: *Starlike Functions, Convex Function, Hankel Determinants.*

1. Introduction

Let consider $\mathcal{K}$ as the class of functions f of the form

$$f(z_1) = z_1 + \sum_{n=2}^{\infty} b_n z_1^n \tag{1}$$

analytic in $E := (z_1 : |z_1| < 1, z_1 \epsilon \overline{C})$ and let $\mathfrak{R} \subset K$ that contain all the univalent functions in E.

Let take two analytic functions as f and j. Then f is subordinated to j and written as $f \prec j$, if Schwarz Lemma exists with $w_1(0) = 0$ then $f(z_1) = j(w_1(z_1))$. In addition to this if j is

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univalent and $f(0) = j(0)$, then $f(E) \subseteq j(E)$. Now we used Starlike class S^{**} which was introduced by Raina and Sakol [15];

$$S^{**} = \{f \epsilon \Re \ |(\frac{z_1 f\prime(z_1)}{f(z_1)})^2 - 1| \leq 2|\frac{z_1 f\prime(z_1)}{f}(z_1)|, z_1 \epsilon E\} \tag{2}$$

Practically, the Starlike function is defined as the class of analytic functions that are compatible with some geometry conditions. They have been extensively studied in the field of complex analysis, and have a wide range of applications in different fields of mathematics, physics or engineering. The study of convexity is one of the practical applications of starlike functions. Convexity is an essential concept in optimization theory, and starlike functions play a significant role in solving convex optimisation problems.

There are also applications for starlike functions when it comes to image processing. For example, they're capable of modeling the shape of objects in an image. In order to get an accurate estimate of the shape, it is possible to create a starlike function on the boundary of this object.

The study of fluid dynamics is a further application of starlike functions. In this case, it is possible to use starlike functions for simulation of fluid flow through obstacles. It is possible to gain insight into the behaviour of fluids in various situations by analysing the properties of these functions.

Geometrically, a function $f \epsilon S^{**}$ is that, for any $z_1 \epsilon E$, the ratio $z_1 \frac{f\prime(z_1)}{f(z_1)}$ contains the region which is bounded by Lune operator. Lune Operator is also a univalent function having normalized form. It is defined by the given relation $\{w_1 \epsilon \overline{C}: |w_1^2 - 1| \leq 2|w_1|\}$.

By using the definitions of subordination property and by using Lune Operator we can defined starlike class as

$$S^{**} := \{f \epsilon \Re \ : \frac{z_1 f\prime(z_1)}{f(z_1)} \prec z_1 + \sqrt{1 + (z_1)^2} = q_1(z_1), z_1 \epsilon E\},$$

here the branch of the square root is taken to be $q_1(0) = 1$. The convex class C_ς related to the function q_1 can be defined as

$$C_\varsigma := \{f \epsilon \Re \ : 1 + \frac{z_1 f\prime\prime(z_1)}{f(z_1)} \prec q_1(z_1), z_1 \epsilon E\} \tag{3}$$

A convex function means that the value at any point in time is no greater than the weighted average of its values for two other points. In various areas of mathematical, scientific, technical

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and economic disciplines, Convex functions have a wide range of useful applications. The use of convex functions and their applications can be seen in some examples:

1. Exponential functions, which model phenomena such as population expansion, radioactive decay and compound interests, are a convex function.

2. Logarithmic functions is a concave function (the negative of a convex function) that measures information entropy, relative change, and elasticity.

3. A convex optimization problem is a problem of minimizing a convex function over a convex set. Convex optimization issues can be solved more easily than general optimisation difficulties, and are used in many areas such as circuit design, controller layout, machine learning or signal processing.

The study of the properties of univalent functions that map unit disks to regions without intersection is one of the useful uses of convex functions in geometric function theory. Convex functions are useful for a number of reasons, i.e. because they satisfy certain inequalityes such as Jensen or Harnacks inequalities which enable us to compare their values across different parts of the function.

Many authors have devoted considerable time to the class S^{**} . Raina and Sokół [22],[23] have studied the coefficient estimates of the class S^{**} while the radius problems of the class S^{**} was investigated by Gandhi and Ravichandran [6]. Sharma et al[17] solved certain differential subordinations related to the class S^{**}. Raina et al[14]. ensured that the functions of a class S^{**} are represented in an integrated way and under sufficient conditions. Cho and Kumar obtained some results on coefficient estimates. Cho et al[5]. settled the coefficient conjecture of this class S^{**}.

In the field of function theory, there has been considerable emphasis on assessing the limits of determinants of Hankel matrices[25],[26] whose coefficients are analytic functions f defined in E of the form (1). Hankel matrices and determinants are a key element in some parts of the mathematics, with different applications[19].

A Hankel determinant, which is a square matrix with constant skew diagonals, has been created by picking up elements from the Hankel matrix. The coefficients of power series expansions of functions are usually the elements of the Hankel matrix or their derivatives in terms of geometry functions theory.

Practical applications of Hankel determinant are as follows:

1. Finding the coefficients and limit of some types of universal functions which map unit disk to regions without individual selfintercepts is one of the useful applications of Hankel's determinants in Geometric Function Theory.

Power System Technology
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2. Using the Hankel matrix singular value decomposition, to find statespace realizations for a given sequence of outputs.

3. Computing the asymptotics of the determinants of some integral operators that arise in random matrix theory and its applications, such as the hard-edge scaling for ensembles of positive Hermitian matrices.

The study of qth Hankel determinant for $f \epsilon K$ was defined by Pommerenke [13], assuming that $b_1 := 1$

$$H_{q,n}(f) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+q-1} \\ b_{n+1} & b_{n+2} & \cdots & b_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n+q-1} & b_{n+q} & \cdots & b_{n+2q-2} \end{vmatrix} \quad (4)$$

here $n \geq 1$ and $q \geq 1$. We obtained Hankel fourth order determinant expression from (4) by puting $n = 1$ and $q = 4$

$$|H_{4,1}(f)| = |b_4^4 - 3b_3 b_4^2 b_5 + b_3^2 b_5^2 + 2b_2 b_4 b_5^2 - b_5^3 + 2b_3^2 b_4 b_6$$
$$-2b_2 b_4^2 b_6 - 2b_2 b_3 b_5 b_6 + 2b_4 b_5 b_6 + b_2^2 b_6^2 - b_3 b_6^2 - b_3^3 b_7 \quad (5)$$
$$+2b_2 b_3 b_4 b_7 - b_4^2 b_7 - b_2^2 b_5 b_7 + b_3 b_5 b_7|$$

Over the past years, many authors [1],[2],[3],[4],[7],[8],[16],[24] found out the sharp coefficient bounds for Hankel fourth order determinant using different results because of Kwon et al [10],[11]. for few subclasses of univalent functions.

Now the class of analytic function R can be defined in term of P series for $z_1 \epsilon E$ as;

$$R(z_1) = 1 + \sum_{n=1}^{\infty} d_n z_1^n \quad (6)$$

where $Re(R(z_1)) \geq 0$ in E.

The formula for $d_i (i = 1,2,3,4,5,6,7)$, that is included in the Lemma (1.1) which is mentioned below, this lemma has a key role to play in determining the sharp bounds for Hankel determinants and are an essential building block of our major results. The relation for $d_i (i = 1,2,3,4,5,6,7)$, is found in 2.

Power System Technology
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2. Objectives

The following objectives has been obtained by this study;

• The behaviour and properties of coefficients has been analyzed within the class of starlike and convex function.

• The upper bounds for the subclass of analytic function have been obtained.

• The sharp coefficient bounds of Hankel fourth order determinant has been found by using schwarz lemma and subordination property.

3. Methods

Lorem ipsum dolor sit amet, [9],[12] If $p\epsilon P$ and is of the form (6) then

$$|d_n| \leq 2, n \geq 1, \tag{7}$$

$$|d_{n+t} - a d_n d_t| \leq \begin{pmatrix} 2, 0 \leq a \leq 1; \\ 2|2a-1|, otherwise, \end{pmatrix} \tag{8}$$

and

$$|d_1^3 - a d_3| \leq \begin{pmatrix} 2|a-4|, a \leq 4/3; \\ 2a\sqrt{\frac{a}{a-1}}, a > 4/3 \end{pmatrix} \tag{9}$$

for some $\eta_1 \epsilon [0,1]$ and $\eta_2, \eta_3, \eta_4, \eta_5 \epsilon E$.

4. Results

Now, we've proven the first major theorem of this paper.

Theorem 2.1. Let $f\epsilon S^{**}$ and given by (1). Then

$$|H_{4,1}(f)| \leq 0.890813 \tag{10}$$

This result is sharp for

$$f_0(z_1) = z_1\left(\int_0^{z_1}(1+\frac{3}{2}t+\frac{5}{4}t^2)dt\right) = z_1^2 + \frac{3}{4}z_1^3 + \frac{5}{12}z_1^4 \ldots, \tag{11}$$

with $f(0) = 0$ and $f'(0) = 1$, acts as an external function for the bound of $|H_{4,1}(f)|$ for $b_2 = 1, b_3 = \frac{3}{4}, b_4 = \frac{5}{12}$.

Power System Technology
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Proof. Let $f\epsilon S^{**}$. [21] Then by using the definition of subordination, it can be written as;

$$\frac{z_1 f\prime(z_1)}{f(z_1)} = w_1(z_1) + \sqrt{1 + w_1^2(z_1)}, \tag{12}$$

here w_1 is a Schwarz function with $w_1(0) = 0$ and $|w_1(z_1)| < 1$ in E. Let $R\epsilon P$. Then we can write

$$w_1(z_1) = \frac{R(z_1)-1}{R(z_1)+1} \tag{13}$$

Now by taking the value of $R(z_1)$ and $w_1(z_1)$ from (6) and (13) respectively and putting in (12), we get the following coefficients, the first four bounds from $b_2 \; to \; b_5$ are same as Riaz et al [21]

$$b_2 = \frac{1}{2} d_1 \tag{14}$$

$$b_3 = \frac{1}{16} d_1^2 + \frac{1}{4} d_2 \tag{15}$$

$$b_4 = \frac{1}{24} d_1 d_2 + \frac{1}{6} d_3 - \frac{1}{96} d_1^3 \tag{16}$$

$$b_5 = \frac{1}{384} d_1^4 - \frac{5}{192} d_2 d_1^2 + \frac{1}{48} d_1 d_3 + \frac{1}{8} d_4 \tag{17}$$

$$b_6 = \frac{3}{128} d_2 d_1^3 - \frac{1}{48} d_1^2 d_3 - \frac{1}{3840} d_1^5 + \frac{1}{10} d_5 + \frac{1}{80} d_1 d_4 - \frac{1}{120} d_2 d_3 - \frac{1}{48} d_1 d_2^2 \tag{18}$$

$$b_7 = \frac{-59}{184320} d_1^6 - \frac{11}{1152} d_1^4 d_2 + \frac{1}{288} d_1^2 d_2^2 - \frac{1}{192} d_2^3 - \frac{1}{2304} d_1^3 d_3 - \frac{97}{2880} d_1 d_2 d_3 - \frac{1}{144} d_3^2 - \frac{11}{640} d_1^2 d_4 - \frac{1}{96} d_2 d_4 + \frac{1}{12} d_6 + \frac{1}{120} d_1 d_5 \tag{19}$$

Now we can figure out the value of Hankel Fourth order determinant by using the following expression

$$H_{4,1}(f) = b_7 H_{3,1}(f) - b_6 ß_1 + b_5 ß_2 - b_4 ß_3 \tag{20}$$

where

$$ß_1 = b_6(b_3 - b_2^2) + b_3(b_2 b_5 - b_3 b_4) - b_4(b_5 - b_2 b_4) \tag{21}$$

Power System Technology
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$$ß_2 = b_3(b_3b_5 - b_4^2) - b_5(b_5 - b_2b_4) + b_6(b_4 - b_2b_3) \quad (22)$$

$$ß_3 = b_4(b_3b_5 - b_4^2) - b_5(b_2b_5 - b_3b_4) + b_6(b_4 - b_2b_3) \quad (23)$$

Now by using Lemma (1.1), we first determine the bounds of $ß_1$ by taking the values of $b_i's(i = 1,2,3,4,5,6,7)$ from (14)-(19) and putting in (21), we got

$$ß_1 = \frac{31}{122880}d_1^7 + \frac{1}{512}d_1^4d_3 + \frac{1}{240}d_1^2d_2d_3 - \frac{43}{7680}d_1^5d_2 + \frac{11}{3840}d_1^3d_4 - \frac{1}{48}d_3d_4 + \frac{1}{128}d_1^3d_2^2 - \frac{1}{80}d_2^2d_3 + \frac{13}{960}d_1d_2d_4 - \frac{1}{128}d_1d_2^3 + \frac{1}{96}d_1d_3^2 + \frac{1}{40}d_2d_5 - \frac{3}{160}d_1^2d_5$$

or

$$|ß_1| \le |d_1^4(\frac{31}{122880}d_1^3 + \frac{1}{512}d_3)| + |d_1^2d_2(\frac{1}{240}d_3 - \frac{43}{7680}d_1^3)| + |d_4(\frac{11}{3840}d_1^3 - \frac{1}{48}d_3)| + |d_2^2(\frac{1}{128}d_1^3 - \frac{1}{80}d_3)| + |d_1d_2(\frac{13}{960}d_4 - \frac{1}{128}d_2^2)| + |\frac{1}{96}d_1d_3^2| + |d_5(\frac{1}{40}d_2 - \frac{3}{160}d_1^2)| \quad (24)$$

Using Lemma (1.1) and the triangle inequality, we got,

$$\left|d_1^4\left(\frac{31}{122880}d_1^3 + \frac{1}{512}d_3\right)\right| \le \frac{91}{960}, \left|d_1^2d_2\left(\frac{1}{240}d_3 - \frac{43}{7680}d_1^3\right)\right| \le \frac{9}{80}, \quad (25)$$

$$\left|d_4\left(\frac{11}{3840}d_1^3 - \frac{1}{48}d_3\right)\right| \le \frac{1}{3}\sqrt{\frac{15}{69}}, \left|d_2^2\left(\frac{1}{128}d_1^3 - \frac{1}{80}d_3\right)\right| \le \frac{1}{3}\sqrt{\frac{8}{3}}, \quad (26)$$

$$\left|d_1d_2\left(\frac{13}{960}d_4 - \frac{1}{128}d_2^2\right)\right| \le \frac{13}{120}, \left|\frac{1}{96}d_1d_3^2\right| \le \frac{1}{12}, \left|d_5\left(\frac{1}{40}d_2 - \frac{3}{160}d_1^2\right)\right| \le \frac{1}{10}. \quad (27)$$

By using equation (24) in view of (25)-(27), we got

$$|ß_1| \le \frac{91}{960} + \frac{9}{80} + \frac{1}{3}\sqrt{\frac{15}{69}} + \frac{1}{5}\sqrt{\frac{8}{3}} + \frac{13}{120} + \frac{1}{20} + \frac{1}{10} \approx 0.980974$$

For finding the bounds of $ß_2$ we used same method as we used for $ß_1$. Now by substituting the values of $b_i's$ (i=1,2,...,7) in (22) from equations (14)-(19), as follows;

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$$ß_2 = -\frac{1}{163840}d_1^8 + \frac{23}{20480}d_1^5 d_3 + \frac{23}{3840}d_1^3 d_2 d_3 - \frac{51}{81920}d_1^6 d_2 - \frac{19}{3840}d_1 d_2^2 d_3 - \frac{35}{12288}d_1^4 d_2^2$$
$$+\frac{7}{960}d_1 d_3 d_4 - \frac{41}{30720}d_1^4 d_4 + \frac{1}{60}d_3 d_5 - \frac{1}{240}d_1^3 d_5 - \frac{1}{3072}d_1^2 d_2^3 + \frac{1}{128}d_2^2 d_4 - \frac{1}{64}d_4^2$$
$$-\frac{1}{256}d_1^2 d_3^2 - \frac{1}{120}d_2 d_3^2 + \frac{23}{1920}d_1^2 d_2 d_4 - \frac{1}{120}d_1 d_2 d_5$$

or

$$|ß_2| \leq |d_1^5(-\frac{1}{163840}d_1^3 + \frac{23}{20480}d_3)| + |d_1^3 d_2(\frac{23}{3840}d_3 - \frac{51}{81920}d_1^3)|$$
$$+|-d_1 d_2^2(\frac{19}{3840}d_3 + \frac{35}{12288}d_1^3)| + |d_1 d_4(\frac{7}{960}d_3 - \frac{41}{30720}d_1^3)|$$
$$+|d_5(\frac{1}{60}d_3 - \frac{1}{240}d_1^3)| + |-\frac{1}{3072}d_1^2 d_2^3| + |d_4(\frac{1}{128}d_2^2 - \frac{1}{64}d_4)| + \quad (28)$$
$$|-d_3^2(\frac{1}{256}d_1^2 + \frac{1}{120}d_2)| + |d_1 d_2(\frac{23}{1920}d_1 d_4 - \frac{1}{120}d_5)|$$

Lemma 1.1 and the triangle inequality lead us to

$$\left|d_1^5\left(-\frac{1}{163840}d_1^3 + \frac{23}{20480}d_3\right)\right| \leq \frac{9}{128}, \left|d_1^3 d_2\left(\frac{23}{3840}d_3 - \frac{51}{81920}d_1^3\right)\right| \leq \frac{23}{15}\sqrt{\frac{23}{1319}}, \quad (29)$$

$$\left|d_1 d_2^2\left(\frac{19}{3840}d_3 + \frac{35}{12288}d_1^3\right)\right| \leq \frac{251}{960}, \left|d_1 d_4\left(\frac{7}{960}d_3 - \frac{41}{30720}d_1^3\right)\right| \leq \frac{7}{30}\sqrt{\frac{14}{83}}, \quad (30)$$

$$\left|d_5\left(\frac{1}{60}d_3 - \frac{1}{240}d_1^3\right)\right| \leq \frac{2}{15}\sqrt{\frac{1}{3}}, \left|\frac{1}{3072}d_1^2 d_2^3\right| \leq \frac{1}{96}, \left|d_4\left(\frac{1}{128}d_2^2 - \frac{1}{64}d_4\right)\right| \leq \frac{1}{16}, \quad (31)$$

$$\left|d_3^2\left(\frac{1}{256}d_1^2 + \frac{1}{120}d_2\right)\right| \leq \frac{1}{15}, \left|d_1 d_2\left(\frac{23}{1920}d_1 d_4 - \frac{1}{120}d_5\right)\right| \leq \frac{7}{120}. \quad (32)$$

By substituting the values from equations (29)-(32) in equation (28)

$$|ß_2| \leq \frac{9}{128} + \frac{23}{15}\sqrt{\frac{23}{1319}} + \frac{251}{960} + \frac{7}{30}\sqrt{\frac{14}{83}} + \frac{2}{15}\sqrt{\frac{1}{3}} + \frac{1}{96} + \frac{1}{16} + \frac{1}{15} + \frac{7}{120} \approx 0.904975.$$

For finding the bounds of $ß_3$ we used same method as we used for $ß_2, ß_1$. Now by substituting the values of $b_i's$ (i=1,2,...,7) in (23) from equations (14)-(19), as follows;

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$$ß_3 = \frac{1}{92160}d_1^8 + \frac{19}{23040}d_1^5 d_3 - \frac{5}{884736}d_1^9 - \frac{1}{12288}d_1^6 d_3 + \frac{13}{147456}d_1^7 d_2 + \frac{1}{1536}d_1^4 d_2 d_3$$
$$+\frac{5}{4608}d_1^3 d_3^2 - \frac{1}{216}d_3^3 + \frac{5}{1536}d_1^3 d_2 d_4 + \frac{1}{96}d_2 d_3 d_4 - \frac{5}{4608}d_1^4 d_2^2 + \frac{1}{60}d_3 d_5 - \frac{1}{240}d_1^3 d_5$$
$$+\frac{23}{3840}d_1^3 d_2 d_3 - \frac{11}{11520}d_1^6 d_2 - \frac{1}{576}d_1 d_2 d_3^2 - \frac{17}{73728}d_1^5 d_2^2 - \frac{1}{384}d_1^2 d_2^2 d_3 + \frac{1}{480}d_1 d_3 d_4$$
$$-\frac{1}{920}d_1^4 d_4 - \frac{1}{2048}d_1^5 d_4 + \frac{1}{384}d_1 d_2^2 d_4 - \frac{1}{128}d_1 d_4^2 - \frac{1}{360}d_1 d_2^2 d_3 - \frac{1}{120}d_1 d_2 d_5 - \frac{1}{720}d_2 d_3^2$$
$$-\frac{1}{288}d_1^2 d_3^2 + \frac{1}{576}d_1^2 d_2^3 - \frac{1}{960}d_1^2 d_2 d_4 - \frac{17}{27648}d_1^3 d_2^3$$
.

or

$$|ß_3| \le |d_1^5(\frac{1}{92160}d_1^3 + \frac{19}{23040}d_3)| + |d_1^6(\frac{5}{884736}d_1^3 + \frac{1}{12288}d_3)|$$
$$+|d_1^4 d_2(\frac{13}{147456}d_1^3 + \frac{1}{1536}d_3)| + |d_3^2(\frac{5}{4608}d_1^3 - \frac{1}{216}d_3)|$$
$$+|d_2 d_4(\frac{5}{1536}d_1^3 + \frac{1}{96}d_3)| + |\frac{5}{4608}d_1^4 d_2^2| + |d_5(\frac{1}{60}d_3 - \frac{1}{240}d_1^3)|$$
$$+|d_1^3 d_2(\frac{23}{3840}d_3 - \frac{11}{11520}d_1^3)| + |\frac{1}{576}d_1 d_2 d_3^2| + |d_1^2 d_2^2(\frac{17}{73728}d_1^3 + \frac{1}{384}d_3)| \quad (33)$$
$$+|d_1 d_4(\frac{1}{480}d_3 - \frac{1}{1920}d_1^3)| + |\frac{1}{2048}d_1^5 d_4| + |d_1 d_4(\frac{1}{384}d_2^2 - \frac{1}{128}d_4)|$$
$$+|d_1 d_2(\frac{1}{360}d_2 d_3 + \frac{1}{120}d_5)| + |d_3^2(\frac{1}{720}d_2 + \frac{1}{288}d_1^2| + |d_1^2(\frac{1}{576}d_2^2 - \frac{1}{960}d_4)|$$
$$+|\frac{17}{27648}d_1^3 d_2^3|$$
.

Lemma 1.1 and the triangle inequality lead us to

$$\left|d_1^5\left(\frac{1}{92160}d_1^3 + \frac{19}{23040}d_3\right)\right| \le \frac{1}{18}, \left|d_1^6\left(\frac{5}{884736}d_1^3 + \frac{1}{12288}d_3\right)\right| \le \frac{23}{1728}, \quad (34)$$

$$\left|d_1^4 d_2\left(\frac{13}{147456}d_1^3 + \frac{1}{1536}d_3\right)\right| \le \frac{37}{576}, \left|d_3^2\left(\frac{5}{4608}d_1^3 - \frac{1}{216}d_3\right)\right| \le \frac{8}{189}, \quad (35)$$

$$\left|d_2 d_4\left(\frac{5}{1536}d_1^3 + \frac{1}{96}d_3\right)\right| \le \frac{3}{16}, \left|\frac{5}{4608}d_1^4 d_2^2\right| \le \frac{5}{72}, \left|d_5\left(\frac{1}{60}d_3 - \frac{1}{240}d_1^3\right)\right| \le \frac{15}{240}, \quad (36)$$

$$\left|d_1^3 d_2\left(\frac{23}{3840}d_3 - \frac{11}{11520}d_1^3\right)\right| \le \frac{23}{120}\sqrt{\frac{69}{58}}, \left|\frac{1}{576}d_1 d_2 d_3^2\right| \le \frac{1}{36}, \quad (37)$$

$$\left|d_1^2 d_2^2\left(\frac{17}{73728}d_1^3 + \frac{1}{384}d_3\right)\right| \le \frac{65}{576}, \left|d_1 d_4\left(\frac{1}{480}d_3 - \frac{1}{1920}d_1^3\right)\right| \le \frac{1}{30}\sqrt{\frac{1}{3}}, \quad (38)$$

Power System Technology
ISSN:1000-3673
Received: 06-04-2024 Revised: 15-05-2024 Accepted: 28-06-2024

$$\left|\frac{1}{2048} d_1^5 d_4\right| \leq \frac{1}{32}, \left|d_1 d_4 \left(\frac{1}{384} d_2^2 - \frac{1}{128} d_4\right)\right| \leq \frac{1}{16}, \left|d_1 d_2 \left(\frac{1}{360} d_2 d_3 + \frac{1}{120} d_5\right)\right| \leq \frac{1}{9}, \quad (39)$$

$$|d_3^2(\frac{1}{720} d_2 + \frac{1}{288} d_1^2| \leq \frac{4}{15}, \left|d_1^2 \left(\frac{1}{576} d_2^2 - \frac{1}{960} d_4\right)\right| \leq \frac{7}{180}, \left|\frac{17}{27648} d_1^3 d_2^3\right| \leq \frac{17}{432}, \quad (40)$$

By substituting the values from equations (34)-(40) in equation (33)

$$|ß_3| \leq \frac{1}{18} + \frac{23}{1728} + \frac{37}{576} + \frac{8}{189} + \frac{3}{16} + \frac{5}{72} + \frac{15}{240} + \frac{23}{120}\sqrt{\frac{69}{58}} + \frac{1}{36} + \frac{65}{576} + \frac{1}{30}\sqrt{\frac{1}{3}} + \frac{1}{32} + \frac{1}{16} + \frac{1}{9} + \frac{4}{15} + \frac{7}{180} + \frac{17}{432} \approx 1.413566.$$

Remark 1.2. We found the bounds of $ß_1$, $ß_2$ and $ß_3$ as 0.980974, 0.904975 and 1.413566 respectively by using the above calculation.

Now we will see the bounds of initial coefficients b_i for $i = 2,3,4,5$. These bounds are already derived in [20][18], these are presented in the form of following remark.

Remark 1.3. For $f \epsilon S^{**}, |b_2| \leq 1, |b_3| \leq 3/4, |b_4| \leq 5/12, |b_5| \leq 1/8$. Now the first three bounds are sharp.

It is significantly more difficult to find coefficient boundaries for n > 5. In order to resolve this issue, we shall use Lemma 1.1 for determining the bounds of 6th and 7th coefficient for functions of the class S^{**} in the following Lemma.

Lemma 1.4. Let $f \epsilon S^{**}$. Then $|b_6| \leq 8/15 \approx 0.5333 and |b_7| \leq 5611/5760 \approx 0.97413.$

Proof: Taking the values of b_6 from the equation (18) and by arranging these values in suitable sequence we got

$$3840 b_6 = [90 d_2 d_1^3 - 80 d_1^2 d_3 - 80 d_1 d_2^2 + 48 d_1 d_4 - d_1^5 - 32 d_2 d_3 + 384 d_5]$$

It can be seen from the perspective of triangle inequality

$$3840|b_6| \leq |d_1^2(98 d_1 d_2 - 80 d_3)| + |d_1(48 d_4 - 80 d_2^2)| + \left|-d_1^5\right| + |384 d_5 - 32 d_2 d_3 \quad (41)$$

Let's look at the following inequalities using Lemma 1.1

Received: 06-04-2024 Revised: 15-05-2024 Accepted: 28-06-2024

$|d_1^2(98d_1d_2-80d_3)| \le 800, |d_1(48d_4-80d_2^2)| \le 448, |-d_1^5| \le 32, |384d_5-32d_2d_3| \le 768$ (42)

By putting above obtained values in eq (1.41), we have

$$3840|b_6| \le 800+448+32+768 \quad (43)$$

$$|b_6| \le \frac{8}{15} \approx 0.5333. \quad (44)$$

Now by applying same procedure for b_7, we have

$$184320b_7 = -59d_1^6 - 1760d_1^4d_2 + 640d_1^2d_2^2 - 960d_2^3 - 80d_1^3d_3 - 23168d_1^2d_4 + 1536d_1d_5 -6208d_1d_2d_3 - 1280d_3^2 - 1920d_2d_4 + 15360d_6$$

As a result of triangle inequality, it can also be viewed as

$$184320|b_7| = |-d_1^4(59d_1^2+1760d_2)| + |d_2^2(640d_1^2-960d_2)| +|d_1^2(80d_1d_3+3168d_4)| + |d_1(1536d_5-6208d_2d_3)| + |-1280d_3^2| + |-1920d_2d_4| + |15360d_6| \quad (45)$$

Lemma (1.1) taking us through the following steps:

$|d_1^4(59d_1^2+1760d_2)| \le 58208, |d_2^2(640d_1^2-960d_2)| \le 7680, |d_1^2(80d_1d_3+3168d_4)| \le 26624$
$|d_1(1536d_5-6208d_2d_3)| \le 43520, |1280d_3^2| \le 5120,$
$|(-1920d_2d_4)| \le 7680, |15360d_6| \le 30720$ (46)

By using the above values, we have

$$184320|b_7| \le 58208+7680+26624+43520+5120+7680+30720$$
$$|b_7| \le \frac{5611}{5760} \approx 0.97413. \quad (47)$$

Power System Technology
ISSN:1000-3673
Received: 06-04-2024 Revised: 15-05-2024 Accepted: 28-06-2024

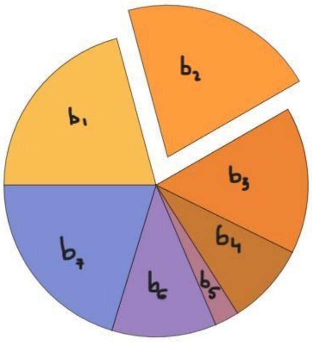

Figure 1: Hankel Coefficients of Starlike function

Theorem 2.2. If $f\epsilon C_{\varsigma}$ is in the form of (1.1). Then

$$|H_{4,1}| \leq 0.00586052 \tag{48}$$

This result is sharp for

$$j_0(z_1) = \int_0^{z_1} z_1\left(\frac{1}{2}+\frac{1}{2}t+\frac{5}{16}t^2dt\right) = \frac{1}{2}z_1^2+\frac{1}{4}z_1^3+\frac{5}{48}z_1^4+\ldots \tag{49}$$

Proof: Let $f\epsilon C_{\varsigma}$. Then by using subordination property, we have

$$1+\frac{z_1f''(z_1)}{f'(z_1)} = w_1(z_1)+\sqrt{1+w_1^2(z_1)} \tag{50}$$

here w_1 is a Schwarz function with $w_1(0)=0$ and $|w_1(z_1)|<1$ in E. Let $R\epsilon P$. Then we can write

$$w_1(z_1) = \frac{R(z_1)-1}{R(z_1)+1} \tag{51}$$

Now by taking the value of $R(z_1)$ and $w_1(z_1)$ from (6) and (51) respectively and putting in (50), we get the following coefficients, , the first four bounds from $b_2 to b_5$ are same as Riaz et al[21]

$$b_2 = \frac{1}{4}d_1 \tag{52}$$

$$b_3 = \frac{1}{48}d_1^2+\frac{1}{12}d_2 \tag{53}$$

$$b_4 = \frac{1}{96}d_1d_2+\frac{1}{24}d_3-\frac{1}{384}d_1^3 \tag{54}$$

Power System Technology
ISSN:1000-3673
Received: 06-04-2024 Revised: 15-05-2024 Accepted: 28-06-2024

$$b_5=\frac{1}{1920}d_1^4-\frac{1}{192}d_2d_1^2+\frac{1}{240}d_1d_3+\frac{1}{40}d_4 \quad (55)$$

$$b_6=\frac{-1}{640}d_1^4+\frac{1}{11520}d_1^5+\frac{23}{11520}d_2d_1^3-\frac{1}{480}d_1d_2^2-\frac{7}{2880}d_1^2d_3+\frac{1}{80}d_2d_3+\frac{1}{480}d_1d_4+\frac{1}{60}d_5 \quad (56)$$

$$b_7=\frac{41}{80640}d_1^5-\frac{107}{10290240}d_1^6-\frac{5}{8064}d_1^3d_2-\frac{19}{53760}d_1^4d_2+\frac{79}{40320}d_1^2d_2^2-\frac{1}{1344}d_2^3+\frac{1}{1008}d_1^2d_3+\frac{1}{840}d_1^3d_3-\frac{1}{105}d_1d_2d_3-\frac{1}{1008}d_3^2+\frac{1}{336}d_1d_4-\frac{11}{4480}d_1^2d_4-\frac{1}{672}d_2d_4+\frac{1}{840}d_1d_5-\frac{1}{84}d_6 \quad (57)$$

Now we can figure out the value of Hankel Fourth order determinant by using the following expression

$$H_{4,1}(f)=b_7H_{3,1}(f)-b_6\xi_1+b_5\xi_2-b_4\xi_3 \quad (58)$$

where

$$\xi_1=b_6(b_3-b_2^2)+b_3(b_2b_5-b_3b_4)-b_4(b_5-b_2b_4) \quad (59)$$

$$\xi_2=b_3(b_3b_5-b_4^2)-b_5(b_5-b_2b_4)+b_6(b_4-b_2b_3) \quad (60)$$

$$\xi_3=b_4(b_3b_5-b_4^2)-b_5(b_2b_5-b_3b_4)+b_6(b_4-b_2b_3) \quad (61)$$

Now by using Lemma (1.1), we first determine the bounds of ξ_1. By taking the values of $b_i's(i=1,2,3,4,5,6,7)$ from (52)-(57) and putting in (59), we got

$$\xi_1=\frac{29}{8847360}d_1^7-\frac{133}{1105920}d_1^5d_2+\frac{23}{110592}d_1^3d_2^2-\frac{17}{69120}d_1d_2^3+\frac{1}{13824}d_1^4d_3+\frac{1}{11520}d_1d_3^2-\frac{1}{2880}d_1^2d_2d_3+\frac{13}{17280}d_2^2d_3+\frac{1}{2304}d_1d_2d_4-\frac{1}{960}d_3d_4+\frac{1}{720}d_2d_5-\frac{1}{1440}d_1^2d_5+\frac{1}{9216}d_1^3d_4+\frac{1}{15360}d_1^6-\frac{1}{7680}d_1^4d_2$$

or

Power System Technology
ISSN:1000-3673
Received: 06-04-2024 Revised: 15-05-2024 Accepted: 28-06-2024

$$|\xi_1| \le |d_1^5(\frac{29}{8847360}d_1^2 - \frac{133}{1105920}d_2)| + |d_1d_2^2(\frac{23}{110592}d_1^2 - \frac{17}{69120}d_2)|$$
$$+|d_1d_3(\frac{1}{13824}d_1^3 + \frac{1}{11520}d_3)| + |d_2d_3(\frac{13}{117280}d_2 - \frac{1}{2880}d_1^2)|$$
$$+|d_4(\frac{1}{2304}d_1d_4 - \frac{1}{960}d_3)| + |d_5(\frac{1}{720}d_2 - \frac{1}{1440}d_1^2)|$$
$$+|\frac{1}{9216}d_1^3d_4| + |d_1^4(\frac{1}{15360}d_1^2 - \frac{1}{7680}d_2)| \quad (62)$$

Using Lemma (1.1) and the triangle inequality, we got,

$$\left|d_1^5\left(\frac{29}{8847360}d_1^2 - \frac{133}{1105920}d_2\right)\right| \le \frac{133}{17280}, \left|d_1d_2^2\left(\frac{23}{110592}d_1^2 - \frac{17}{69120}d_2\right)\right| \le \frac{17}{8640}, \quad (63)$$

$$\left|d_1d_3\left(\frac{1}{13824}d_1^3 + \frac{1}{11520}d_3\right)\right| \le \frac{13}{4320}, \left|d_2d_3\left(\frac{13}{117280}d_2 - \frac{1}{2880}d_1^2\right)\right| \le \frac{13}{2160}, \quad (64)$$

$$\left|d_4\left(\frac{1}{2304}d_1d_4 - \frac{1}{960}d_3\right)\right| \le \frac{1}{240}, \left|d_5\left(\frac{1}{720}d_2 - \frac{1}{1440}d_1^2\right)\right| \le \frac{1}{180}, \quad (65)$$

$$\left|\frac{1}{9216}d_1^3d_4\right| \le \frac{1}{576}, \left|d_1^4\left(\frac{1}{15360}d_1^2 - \frac{1}{7680}d_2\right)\right| \le \frac{1}{240}, \quad (66)$$

By using equation (62) in view of (63)-(66), we got

$$|\xi_1| \le \frac{133}{17280} + \frac{17}{8640} + \frac{13}{4320} + \frac{13}{2160} + \frac{1}{240} + \frac{1}{180} + \frac{1}{576} + \frac{1}{240} \approx 0.0343$$

For finding the bounds of ξ_2 we used same method as we used for ξ_1. Now by substituting the values of $b_i{}'s$ (i=1,2,...,7) in (60) from equations (52)-(57), as follows;

$$\xi_2 = -\frac{17}{58982400}d_1^8 - \frac{11}{1769472}d_1^6d_2 - \frac{127}{2211840}d_1^4d_2^2 - \frac{13}{552960}d_1^2d_2^3 + \frac{61}{2764800}d_1^5d_3 - \frac{83}{691200}d_1^2d_3^2$$
$$+\frac{13}{138240}d_1^3d_2d_3 - \frac{1}{4320}d_1d_2^2d_3 + \frac{1}{2560}d_1^2d_2d_4 - \frac{11}{230400}d_1^4d_4 + \frac{1}{5760}d_2^2d_4 - \frac{1}{1600}d_4^2 + \frac{1}{1440}d_3d_5$$
$$-\frac{1}{5760}d_1d_2d_5 - \frac{1}{14400}d_1d_3d_4 - \frac{1}{7680}d_1^3d_5 + \frac{13}{34560}d_2d_3^2 + \frac{1}{61440}d_1^5d_2 + \frac{1}{81920}d_1^7 - \frac{1}{15360}d_1^4d_3$$

or

Power System Technology
ISSN:1000-3673
Received: 06-04-2024 Revised: 15-05-2024 Accepted: 28-06-2024

$$|\xi_2| \le |d_1^6(\frac{71}{58182400}d_1^2+\frac{11}{1769472}d_2)|+|d_1^2d_2^2(\frac{127}{2211840}d_1^2+\frac{13}{552960}d_2)|$$
$$+|d_1^2d_3(\frac{61}{2764800}d_1^3-\frac{83}{691200}d_3)|+|d_1d_2d_3(\frac{13}{138240}d_1^2-\frac{1}{4320}d_2)|+|d_1^2d_4(\frac{1}{2560}d_2-\frac{11}{230400}d_1^2)|$$
$$+|d_4(\frac{1}{5760}d_2^2-\frac{1}{1600}d_4)|+|d_5(\frac{1}{1440}d_3-\frac{1}{5760}d_1d_2)|+|-\frac{1}{1440}d_1d_3d_4|+|-\frac{1}{7680}d_1^3d_5|$$
$$+|\frac{13}{34560}d_2d_3^2|+|\frac{1}{61400}d_1^5d_2|+\left|d_1^4\left(\frac{1}{81920}d_1^3-\frac{1}{15360}d_3\right)\right| \tag{67}$$

Lemma 1.1 and the triangle inequality lead us to

$$\left|d_1^6\left(\frac{71}{58182400}d_1^2+\frac{11}{1769472}d_2\right)\right| \le \frac{763}{691200}, \left|d_1^2d_2^2\left(\frac{127}{2211840}d_1^2+\frac{13}{552960}d_2\right)\right| \le \frac{17}{3840}, \tag{68}$$

$$\left|d_1^2d_3\left(\frac{61}{2764800}d_1^3-\frac{83}{691200}d_3\right)\right| \le \frac{83}{21600}\sqrt{\frac{83}{271}}, \left|d_1d_2d_3\left(\frac{13}{138240}d_1^2-\frac{1}{4320}d_2\right)\right| \le \frac{1}{270}, \tag{69}$$

$$\left|d_1^2d_4\left(\frac{1}{2560}d_2-\frac{11}{230400}d_1^2\right)\right| \le \frac{1}{160}, \left|d_4\left(\frac{1}{5760}d_2^2-\frac{1}{1600}d_4\right)\right| \le \frac{1}{400}, \left|d_5\left(\frac{1}{1440}d_3-\frac{1}{5760}d_1d_2\right)\right| \le \frac{1}{360}, \tag{70}$$

$$\left|\frac{1}{1440}d_1d_3d_4\right| \le \frac{1}{1800}, \left|\frac{1}{7680}d_1^3d_5\right| \le \frac{1}{480}, \left|\frac{13}{34560}d_2d_3^2\right| \le \frac{13}{4320}, \tag{71}$$

$$\left|\frac{1}{61400}d_1^5d_2\right| \le \frac{8}{7675}, \left|d_1^4\left(\frac{1}{81920}d_1^3-\frac{1}{15360}d_3\right)\right| \le \frac{1}{80}\sqrt{\frac{1}{3}}. \tag{72}$$

By substituting the values from equations (68)-(72) in equation (67)

$$|\xi_2| \le \frac{763}{691200}+\frac{17}{3840}+\frac{83}{21600}\sqrt{\frac{83}{271}}+\frac{1}{270}+\frac{1}{160}+\frac{1}{400}+\frac{1}{360}+\frac{1}{1800}+\frac{1}{480}+\frac{13}{4320}$$
$$+\frac{8}{7675}+1/80\sqrt{\frac{1}{3}} \approx 0.0341.$$

For finding the bounds of ξ_3 we used same method as we used for ξ_2, ξ_1. Now by substituting the values of $b_i's$ (i=1,2,...,7) in (61) from equations (52)-(57), as follows;

Power System Technology
ISSN:1000-3673
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$$\xi_3=-\frac{1}{1474560}d_1^8-\frac{73}{4423680}d_1^6d_2+\frac{121}{70778880}d_1^7d_2-\frac{151}{1415577600}d_1^9-\frac{1}{221184}d_1^4d_2^2+\frac{1}{46080}d_1^2d_2^3-\frac{89}{17694720}d_1^5d_2^2-\frac{1}{98304}d_1^3d_2^3+\frac{17}{737280}d_1^4d_2d_3-\frac{89}{29491200}d_1^6d_3+\frac{1}{92160}d_1^3d_2d_3-\frac{1}{4608}d_1d_2^2d_3-\frac{7}{69120}d_1^2d_3^2+\frac{1}{1920}d_2d_3^2+\frac{59}{5529600}d_1^3d_3^2+\frac{1}{276480}d_1d_2d_3^2+\frac{1}{11520}d_1d_3d_4-\frac{1}{61440}d_1^4d_4+\frac{1}{15360}d_1^3d_2d_4-\frac{17}{1843200}d_1^5d_4+\frac{1}{23040}d_1d_2^2d_4-\frac{1}{6400}d_1d_4+\frac{1}{5760}d_2d_3d_4-\frac{7}{115200}d_1^2d_3d_4+\frac{1}{1440}d_3d_5-\frac{1}{5760}d_1d_2d_5+\frac{5}{221184}d_1^5d_3-\frac{13}{368640}d_1^2d_2^2d_3-\frac{1}{13824}d_3^3-\frac{1}{46080}d_1^2d_2d_4-\frac{1}{7680}d_1^3d_5+\frac{1}{81920}d_1^7+\frac{1}{61440}d_1^5d_2-\frac{1}{15360}d_1^4d_3$$

or

$$|\xi_3|\le\left|d_1^6\left(\frac{1}{1474560}d_1^2+\frac{73}{4423680}d_2\right)\right|+\left|d_1^7\left(\frac{121}{70778880}d_2-\frac{151}{1415577600}d_1^2\right)\right|+\left|d_1^2d_2^2\left(-\frac{1}{221184}d_1^2+\frac{1}{46080}d_2\right)\right|+\left|d_1^3d_2^2\left(-\frac{89}{17694720}d_1^2-\frac{1}{98304}d_2\right)\right|+\left|d_1^4d_3\left(\frac{17}{737280}d_2-\frac{89}{29491200}d_1^2\right)\right|+\left|d_1d_2d_3\left(\frac{1}{92160}d_1^2-\frac{1}{4608}d_2\right)\right|+\left|d_3^2\left(-\frac{7}{69120}d_1^2+\frac{1}{1920}d_2\right)\right|+\left|d_1d_3^2\left(\frac{59}{5529600}d_1^2+\frac{1}{276480}d_2\right)\right|+\left|d_1d_4\left(\frac{1}{11520}d_3-\frac{1}{61440}d_1^3\right)\right|+\left|d_1^3d_4\left(\frac{1}{15360}d_2-\frac{17}{1843200}d_1^2\right)\right|+\left|d_1d_4\left(\frac{1}{23040}d_2^2-\frac{1}{6400}d_4\right)\right|+\left|d_3d_4\left(\frac{1}{5760}d_2-\frac{7}{115200}d_1^2\right|+\left|d_5\left(\frac{1}{1440}d_3-\frac{1}{5760}d_1d_2\right)\right|+\left|\frac{5}{221184}d_1^5d_3\right|+\left|-\frac{13}{368640}d_1^2d_2^2d_3\right|+\left|-\frac{1}{13824}d_3^3\right|+\left|-\frac{1}{46080}d_1^2d_2d_4\right|+\left|-\frac{1}{7680}d_1^3d_5\right|+\left|\frac{1}{81920}d_1^7\right|+\left|d_1^4\left(\frac{1}{61400}d_1d_2-\frac{1}{15360}d_3\right|\right. \tag{73}$$

Lemma 1.1 and the triangle inequality lead us to

$$\left|d_1^6\left(\frac{1}{1474560}d_1^2+\frac{73}{4423680}d_2\right)\right|\le\frac{97}{34560},\left|d_1^7\left(\frac{121}{70778880}d_2-\frac{151}{1415577600}d_1^2\right)\right|\le\frac{121}{276480}, \tag{74}$$

$$\left|d_1^2d_2^2\left(-\frac{1}{221184}d_1^2+\frac{1}{46080}d_2\right)\right|\le\frac{1}{1440},\left|d_1^3d_2^2\left(-\frac{89}{17694720}d_1^2-\frac{1}{98304}d_2\right)\right|\le\frac{179}{138240}, \tag{75}$$

$$\left|d_1^4d_3\left(\frac{17}{737280}d_2-\frac{89}{29491200}d_1^2\right)\right|\le\frac{17}{11520},\left|d_1d_2d_3\left(\frac{1}{92160}d_1^2-\frac{1}{4608}d_2\right)\right|\le\frac{1}{288},\left|d_3^2\left(-\frac{7}{69120}d_1^2+\frac{1}{1920}d_2\right)\right|\le\frac{1}{240}, \tag{76}$$

Power System Technology
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$$\left|d_1 d_3^2\left(\frac{59}{5529600}d_1^2+\frac{1}{276480}d_2\right)\right|\leq\frac{23}{57600},\left|d_1 d_4\left(\frac{1}{11520}d_3-\frac{1}{61440}d_1^3\right)\right|\leq\frac{1}{360}\sqrt{\frac{1}{3}}, \quad (77)$$

$$\left|d_1^3 d_4\left(\frac{1}{15360}d_2-\frac{17}{1843200}d_1^2\right)\right|\leq\frac{1}{480},\left|d_1 d_4\left(\frac{1}{23040}d_2^2-\frac{1}{6400}d_4\right)\right|\leq\frac{1}{800},|d_3 d_4(\frac{1}{5760}d_2-\frac{7}{115200}d_1^2|\leq\frac{4}{2865} \quad (78)$$

$$\left|d_5\left(\frac{1}{1440}d_3-\frac{1}{5760}d_1 d_2\right)\right|\leq\frac{1}{360},\left|\frac{5}{221184}d_1^5 d_3\right|\leq\frac{5}{3456},\left|\frac{13}{368640}d_1^2 d_2^2 d_3\right|\leq\frac{13}{11520},\left|\frac{1}{13824}d_3^3\right|\leq\frac{1}{1728} \quad (79)$$

$$\left|\frac{1}{46080}d_1^2 d_2 d_4\right|\leq\frac{1}{2880},\left|\frac{1}{7680}d_1^3 d_5\right|\leq\frac{1}{480},\left|\frac{1}{81920}d_1^7\right|\leq\frac{1}{640},|d_1^4(\frac{1}{61400}d_1 d_2-\frac{1}{15360}d_3|\leq\frac{1}{480}. \quad (80)$$

By substituting the values from equations (74)-(80) in equation (73)

$$|\xi_3|\leq\frac{97}{34560}+\frac{121}{276480}+\frac{1}{1440}+\frac{179}{138240}+\frac{17}{11520}+\frac{1}{288}+\frac{1}{240}+\frac{23}{57600}$$
$$+\frac{1}{360}\sqrt{\frac{1}{3}}+\frac{1}{480}+\frac{1}{800}+\frac{4}{2865}+\frac{1}{360}+\frac{5}{3456}+\frac{13}{11520}+\frac{1}{1728}+\frac{1}{2880}+\frac{1}{480}$$
$$+\frac{1}{640}+\frac{1}{480}\approx 0.0331.$$

Remark 2.3. We found the bounds of ξ_1 , ξ_2 and ξ_3 as 0.0343, 0.0341 and 0.0331 respectively by using the above calculation.

Now we will see the bounds of initial coefficients b_i for $i=2,3,4,5$. These bounds are already derived in [22], these are presented in the form of following remark.

Remark 2.4. For $f\epsilon C_{\varsigma}$, $|b_2|\leq 1/2, |b_3|\leq 1/4, |b_4|\leq 5/48, |b_5|\leq 1/30$. Now the first three bounds are sharp.

It is significantly more difficult to find coefficient boundaries for n > 5. In order to resolve this issue, we shall use Lemma 1.1 for determining the bounds of 6th and 7th coefficient for functions of the class C_{ς} in the following Lemma.

Lemma 2.5. Let $f\epsilon C_{\varsigma}$. Then $|b_6|\leq 5/36\approx 0.139 and |b_7|\leq 3533/20160\approx 0.1752$.

Power System Technology
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Proof: Taking the values of b_6 from the equation (56) and by arranging these values in suitable sequence we got

$$11520b_6 = [23d_1^3d_2 - 28d_1^2d_3 - 24d_1d_2^2 + 24d_1d_4 + d_1^5 + 144d_2d_3 + 192d_5 - 18_1^4]$$

It can be seen from the perspective of triangle inequality

$$11520|b_6| \leq |d_1^2(23d_1d_2 - 28d_3)| + |d_1(424d_4 - 24d_2^2)| + |d_1^5| + |192d_5 + 144d_2d_3| + |-18d_1^4|. \quad (81)$$

Let's look at the following inequalities using Lemma 1.1

$$|d_1^2(23d_1d_2 - 28d_3)| \leq 224, |d_1(424d_4 - 24d_2^2)| \leq 96, |d_1^5| \leq 32, |192d_5 + 144d_2d_3| \leq 960, |18d_1^4| \leq 288 \quad (82)$$

By putting above obtained values in (81), we have

$$11520|b_6| \leq 224 + 96 + 32 + 960 + 288 \quad (83)$$

$$|b_6| \leq \frac{5}{36} \approx 0.139. \quad (84)$$

Now by applying same procedure for b_7, we have

$$1290240b_7 = -107d_1^6 - 456d_1^4d_2 + 2528d_1^2d_2^2 - 12288d_1d_2d_3 + 1536d_1^3d_3 - 1280d_3^2 + 1536d_1d_5 -3168d_1^2d_4 - 1920d_2d_4 - 960d_2^3 - 15360d_5 + 3840d_1d_4 + 1280d_1^2d_3 - 800d_1^3d_2 + 656d_1^5$$

As a result of triangle inequality, it can also be viewed as

$$1290240|b_7| = |d_1^4(107d_1^2 + 456d_2)| + |d_1d_2(2528d_1d_2 - 12288d_3)| + |d_3(1536d_1^3 - 1280d_3)| +|d_1(1536d_5 - 3168d_1d_4)| + |-d_2(1920d_4 + 960d_2^2| + |(15360d_5 - 3840d_1d_4)| +|d_1^2(1280d_3 - 800d_1d_2| + |656d_1^5| \quad (85)$$

Lemma (1.1) taking us through the following steps:

$$|d_1^4(107d_1^2 + 456d_2)| \leq 21440, |d_1d_2(2528d_1d_2 - 12288d_3)| \leq 98304,$$
$$|d_3(1536d_1^3 - 1280d_3)| \leq 19456, |d_1(1536d_5 - 3168d_1d_4)| \leq 9600, |d_2(1920d_4 + 960d_2^2| \leq 15360,$$
$$|15360d_5 - 3840d_1d_4| \leq 30720, |d_1^2(1280d_3 - 800d_1d_2| \leq 10240, |656d_1^5| \leq 20992$$

Power System Technology
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(86)

By using the above values, we have

$$1290240|b_7| \leq 21440 + 98304 + 19456 + 9600 + 15360 + 30720 + 10240 + 20992$$
$$|b_7| \leq \frac{3533}{20160} \approx 0.1752.$$

(87)

Hankel coefficients of Convex Function

$b_1 = 1, b_2 = 1/2, b_3 = 1/4, b_4 = 5/48, b_5 = 1/30, b_6 = 5/36, b_7 = 3533/20160$

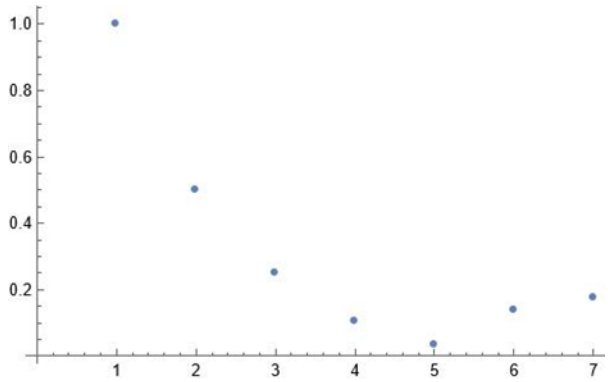

Figure 2: Hankel Coefficients of Convex Function

Potential Application.

1.In the context of univalent functions, which are analytic functions that map a unit disk to another domain, Hankel determinants and coefficient problems have been extensively studied. Applications include the study of bounds for coefficients, extremals and geometric properties related to singlevalent functions that have an impact on conformal mapping, complexity equations and mathematics.

2.In the study of conformal maps, which preserve angles and local shapes, it is useful to understand the behaviour of Hankel determinants and coefficient problems. Applications include the development of accurate maps in cartography, fluid dynamics simulations, and mathematical modeling of physical phenomena

3.The system identification and modelling are based on Hankel determinants and Correlations problems. Applications include estimation of parameters for signal processing, control theory and time series analysis when the model's coefficients are indicative of system dynamics.

Power System Technology
ISSN:1000-3673
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5. Discussion

"In this paper, we have investigated the determinants of Hankel's fourth order for the functions such as convex and starlike by using Lune operator. We have defined a new subclass of normalized analytic functions. With many characteristics and properties, we have succeeded in obtaining sharp coefficients bounds for convex and starlike function. The main results were presented, which have been significantly more difficult to find coefficients for n>5. Finaly the bounds of the 6th and 7th coefficients have also been defined. In addition, we compared our results with those of Amina et al [21]. who has found the third determinant of Hankel for convex and starlike function. We have shown that our results are generalized the previous result and the bounds we got in this study are sharp."

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