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Commutativity Results for Rings with Potent K-Engel Elements

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Abstract:-

Let R be a ring. First we show that if : (P) For each a, b ∈R there exist positive integers n = n(a, b) > 1, m = m(a, b) ≥ 1 and k = k(a, b) ≥ 1, such that [amb, a]n = [amb, a]k, and Nr(R) = 0, then R is commutative. Furthermore, we prove that any s-unital ring satisfying(P)must have a nil commutator ideal. Finally we show that an s-unital ring R is commutative if: (P ′) For each a,b ∈ R there exists positive integer n = n(a,b) > 1 such that (b−br)([xmy, a]−[amb, a]n) = 0, where r > 1 and m ≥ 1 are fixed positive integers, and (Ir) the commutator ideal of R is r!-torsion free.

Keywords: Ring, commutator ideal, s-unital, nil radical, Jacobson radical, potent, k-engel, semi-simple, nilpotent, commutative

1. Introduction

For an associative ring R, let J(R), N (R) stand for the Jacobson radical, the nil radical of R respectively. Also, let Nr(R) denote the sum of all nil right ideals of R. It is wellknown that Nr(R) is thus a two-sided ideal of R. Also C(R) denote the center of

any a ∈ R, is said to be a potent element if a = an for some integern > 1.If (ai)i∈N is a sequence of elements of R and k is a positive integer, we define [a1, a2, . . . , ak+1] inductively as follows:[a1, a2] = a1a2 − a2a1

[a1, . . . , ak, ak+1] = [[a1, . . ,ak], ak+1)If a1 = a and a2 = · · · = ak+1 = b, we write [a1, . . . , ak+1] = [a, b]k, and [a, b]k is called an k-engel element.

A ring R is called left (resp. right) s-unital if for each a∈R, a∈ Ra ∩ aR. If R is an s-unital ring, then for any finite subset F of R, there exists an element e ∈ R such that ea = xa= a for all a∈F (see [5]).

Well-known results due to Jacobson and Herstein assert that a ring

R with any of the following conditions must be commutative.

(i) For every a∈R, there exists an integer n = n(a) > 1 such that an = a, and (ii) For every a, b∈R,there exists an integer n = n(a,b) > 1 such that [a, b]n = [a, b]. Recently it has been proved that [6], if for each a,b ∈ R there exists an integer n = n(a,b) > 1 such that (ab)n = (ab), then R is commutative. Also, Giamburno, Goncalves and Mandel, in [2, Theorem 2.7] have shown that if Nr(R) = 0, then R is commutative if for every a,b∈R, there exist integers n = n(a,b) > 1, m = m(a,b) ≥ 1 and k = k(a,b) ≥ 1 such that [a, bm]n = [a, bm]k.

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In this paper, we extend the results of [6] by showing that if R is a left s-unital ring and for every a,b ∈R, there exist integers n = n(a,b) > 1 and m = m(a,b) ≥ 1 such that (am[b, a])n = am[b,a], then R is commutative.

In order to prove and be able to generalize this result, in Section 1, we investigate the commutativity behavior of the rings satisfying the following condition:

(P) For each a,b ∈ R there exist positive integers n = n(a,b) > 1, m(a,b) ≥ 1 and k = k(a,b) ≥ 1, such that [amb, a]n = [amb,ₖa]k,

and we get some modification of the results of [2].

Finally in Section 2, combining the Jacobson's condition with a spe- cial case of condition (P), we prove that:

Any left s-unital ring is commutative if for fixed integers r > 1 and m ≥ 1 we have

(P ′) For each a,b ∈ R there exists a positive integer n = n(a,b) > 1 such that (b − br)([amb, a] − [amb, a]n) = 0;

and

(Ir) The commutator ideal of R is r!-torsion free.

2. Rings with Potent k-Engel Elements

In preparation for the proof of the main theorems we start with following lemmas. Lemma 2 is obvious and the proof of Lemma 3 can be found in [5].

Lemma 1. Let R be a division ring with the center C. If for each a,b ∈ R there exist positive integers m = m(a,b) ≥ 1 and k = k(a,b) ≥ 1 such that [amb, a]k = 0, then R is commutative.

Proof. Since [amb, a]k = am[b, a]k, this lemma is an immediate consequence of [4]. □

Lemma 2. Let b ∈ R and a ∈ J(R). If ba = b, then b = 0.

Lemma 3. Let R be a left s-unital ring. If for e, a,b ∈ R, ea = a, eb = b and amb = (a + e)mb = 0, then b = 0.

Theorem 1. Let R be a division ring with center C which satisfies condition (P). Then R is commutative.

Proof. Suppose that R is not commutative, and let a∈ R − C. Then by Lemma 1 there exists b ∈ R − C such that, for all integers m ≥ 1 and k ≥ 1, [amb, a]k+1 /= 0, i.e., [amb, a]k

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/∈ C. But by (P), there exist integers m0, k0, n + 1 such that [am0 b, a]n+1 = [am0 b, a]k . Therefore,

k0 0

u = [am0 b, a]k0 is an n-th root of unity which is not in C.

Let r be a positive integer relatively prime to n. By the Noether- Skolem theorem there exists b ∈ R∗ such that, the inner automorphism

ψ(u) = bub−1 restricted to C(u) gives ψ : C(u) → C(u) such that ψ(u) = ur. Since dimC C(u) < ∞, there exists a positive integer l such that ψl = 1. Now let P be the prime field of C. As it is shown in theorem in [2] we can assume that R is finite-dimensional over C.

We claim that P is finite and bl is algebraic over P .

Suppose not, we can construct a nontrivial nonarchimedean valua- tion on C. Assuming R → Mt(C), where Mt(C) is the full t × t matrix ring over C. We can define the norm

$\|A\| = \max_{i,j} |a_{ij}|$, for $A = a_{ij} \in Mt(C)$

whose restriction to C is | |.

Again, as we started the proof, assume that a,b are element of R such that, for every m ≥ 1 and k ≥ 1, [amb, a]k /∈ C. Now we can choose λ ∈ C such that |λ|‖a‖ < 1 and |λ|‖b‖‖a‖ < 1. Let us set a = λa. But ‖ba‖ ≤ ‖b‖‖xa‖ = ‖b‖‖λa < 1. Therefore by an easy computation we have

‖[b,a)‖ < 1 .(1.1)

Hence, if k > 1, m ≥ 1 and n + 1 > 1 are integers depending on x^{-}, y

such that [xamb, xa]n = 1_k in view of (1.1) we have

1 > ‖xa‖m‖[b, xa]k‖ = ‖[xamb, xa]k‖ = 1 .

This contradiction shows that R is a finite noncommutative division ring, which is contradiction.

□

Now we prove the main theorem for semisimple rings and prime rings with charR = p and N (R) = 0.

Theorem 2. Let R be a ring satisfying (P). Then R is commutative, under any of the following conditions:

(i) R is semisimple. (ii) R is prime, charR = p > 0 and N (R) = 0.

Proof. Suppose that R satisfies condition (i). In this case either R ≈ D, for some division ring D-in which case would deduce that R is commutative by use of Theorem 1, or for some k > 1, Dk is a ho- momorphic image of a subring of R. We wish to show that this latter possibility does not arise. If it did, Dk as a homomorphic image of a

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subring of R would inherit the property (P). This is seen to be patently false by considering the elements a = E11 and y = E12, for these satisfy

[amb, a]k = ±E12 = ±y , for any m ≥ 1 and k ≥ 1.

Thus, if R is semisimple it must be commutative.

Now let R satisfies condition (ii). Let x, y ∈ R, then by (P) there exist positive integers k = k(a,b) ≥ 1, m(a,b) ≥ 1 and n = n(a,b) > 1 such that

[amb, a]k =[amb, a]n

=[amb, a]k[amb, a]n−1.

But part (i) shows that R/J(R) is commutative, hence [amb, a]n−1 ∈ J(R) and therefore by Lemma 2, [amb, a]k = 0. Since charR = p > 0 this implies that, for each a, b, r ∈ R there are positive integers

Theorem 3. Let R be a ring satisfying condition (P). If Nr(R) = 0
then R is commutative.

Proof. We claim that R has no nonzero nilpotent elements. Let a ∈ R be such that a2 = 0. Let x ∈ R, k = k(ax, a) ≥ 1, m = m(ax, a) ≥ 1 and n = n(ax, a) > 1 be such that

[(ax)ma, ax]n = [(ax)ma, ax]k.

we have

$$(-1)^k (ax)^{m+k} a = [(ax)^m a, ax]_k$$

$$= [(ax)^m a, ax]^n$$

= (−1)kn((ax)m+ka)n
= 0 ,
i.e. (−1)k(ax)k+m+1 = 0.

Therefore aR is a nil right ideal and since Nr(R) = 0, it follows that
a = 0. Therefore R has no nilpotent elements, as desired.

In view of [4, Theorem 12.7], and the fact that condition (P) holds for homomorphic images, we may assume that R is a domain. If J(R) = 0 or charR = p > 0, then the commutativity of R follows from Theorem
2, so we may assume that J(R) 0 and charR = 0. Repeating the
argument of Theorem 2 (ii) easily shows that for each a,b ∈ R,
am[b,a]k = [amb, a]k = 0
for some m ≥ 1 and k ≥ 1, which implies that [b,a]k = 0, since R is a domain. Therefore R is a commutative ring, by [1, Theorem 3].

□

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We can now prove the main result of this section as follows:

Theorem 4. Let R be a left s-unital ring, which satisfies condition (P), then the commutator ideal of R is nil.

Proof. Repeating the argument of Theorem 2 (ii) shows that for each a,b ∈ R, am[b, a]k = [amb, a]k = 0 for some m ≥ 1 and k ≥ 1. Now replacing a by a + e, where e ∈ R is such that ea = ae = a and eb = be = b, we conclude that

(a + e)m[b, a]k = 0 = am[b, a]k

and thus [b,a]k = 0 by Lemma 3. Therefore the commutator ideal of R is nil, by [1]. □

Corollary 1. Let R be a left s-unital ring, and for each a,b ∈ R there exist m = m(a,b) ≥ 1 and n = n(a,b) > 1 such that (am[b, a])n = am[b,a], then R is commutative.

Proof. By letting k = 1 in (P), the proof of Theorem 4 implies that R is commutative. □

3. Commutativtiy Theorems and Conclusive remarks

Theorem 5. Let R be a division ring. If for each a,b ∈ R there exist positive integers r = r(a,b) > 1, n = n(a,b) ≥ 1, m = m(a,b) ≥ 1 and k = k(a,b) ≥ 1 such that

(b − br)([amb, a]k − [amb, a]n) = 0 . (2.1)

Then R is commutative.

Proof. By Theorem 1, it is enough to show that if there exist a,b ∈ R such that

[amb, a] [amb, a]n for all n ≥ 1 , (2.2)

Theorem 6. Let R be an s-unital ring and r > 1 and m ≥ 1 be fixed positive integers. If R satisfies condition (P ′) and (Ir), then R is commutative.

Proof. We prove Theorem 6, by dividing its proof into several steps.

Step 1. Clearly Theorem 6 is true for any division ring (by Theorem 5).

Step 2. Theorem 6 is true for any semisimple ring.

As in the proof of Theorem 2 (i) choosing a = E22 and y = E21, we see that

(b − br)([amb, a] − [amb, a]n) = −E12 /= 0,

Step 3. Any ring R satisfying the hypotheses of Theorem 6 is com- mutative. Let a,b ∈ R, then by (P ′) we have

(b − br)[amb, a] = (b − br)[amb, a]n

= (b − br)[amb, a][amb, a]n−1.

In view of Step 2, [amb, a]n−1 ∈ J(R), therefore

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(b − br)[amb,a] = 0, (2.4)

by Lemma 2. Since R is an s- unital ring we can replace b by b+e in (2.4), where e ∈ R, ea = ae = a, ye = eb = b. As in the proof of [7, Theorem 3], we can deduce that r![amb, a] = 0 and therefore [amb, a] = 0 by (Ir).

Now replacing a by a+ e we conclude that

(a + e)m[b,a] = 0 = am[b,a]

and thus [b,a] = 0 by Lemma 3. Therefore R is commutative, as desired

□

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